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Divisor Graph and Factorization Type

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Abstract

In this project, we examine some natural ideal conditions and show how graphs can be defined that give a visualization of these conditions. We examine the interplay between the multiplicative ideal theory and the graph-theoretic structure of the associated graph. In this research, we associate a graph in a natural way with the divisors of a commutative ring.

Dedication

To Mehdi and Dion,

Mehdi, your love and support have been my guiding light. Dion, your presence fills my heart with joy every day. This research is dedicated to both of you with all my love.

Masoum

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Contents

Motivation and Background

The concept of "zero divisor graph" of a commutative ring was introduced by D. Anderson and P. Livingston in [4] but arguably had as its genesis some of the ideas in [6]. Let R be a commutative ring with 1 and let $Z(R)$ be the set of nonzero zero divisors (note that $Z(R)$ is empty if R is a domain). We associate a simple graph to R as follows. Let the vertex set be the nonzero zero divisors and declare that for distinct $x, y \in Z(R)$, the vertices x and y are adjacent if and only if $xy = 0$. In [4], Anderson and Livingston introduced and studied the zero divisor graph and this graph turns out to best highlights a number of interesting properties of the zero divisors of a commutative ring. The zero divisor graph helps us to study the algebraic properties of rings using graph theoretical tools. We can translate some algebraic properties of a ring to graph theory language and then the geometric properties of graphs help us to explore some interesting results in the algebraic structures of rings. The zero divisor graph of a commutative ring has been studied extensively by too many authors to list completely, but the interested reader is invited to consult [5], [2], [3], and [1] among others.

In 2000 Singh and Santhosh [12] defined the concept of a divisor graph in the setting of the ordinary integers. They defined a divisor graph G as an ordered pair $(V; E)$ where $V \subsetneq \mathbb{Z}$ and for all $u, v \in V, u \neq v, uv \in E$ if and only if $u \mid v$ or $v \mid u$. Singh and Santhosh showed no odd cycle of length five or more is a divisor graph while all even cycles, complete graphs are divisor graphs. In 2001, Chartrand, Muntean, Saenpholphant and Zhang [7] and also studied divisor graphs. They let S be a finite, nonempty set of positive integers. Then, the divisor graph $G(S)$ of S has S as its vertex set, and vertices i and j are adjacent if and only if either $i \mid j$ or $j \mid i$.

Given this history, we endeavor at first to generalize these notions to a more general setting, and then, inspired by this work, we will look at "factorization types" for elements in monoids and domains. This project is outlined as follows. First, some definitions, theorems, and examples are listed. Proofs that provide useful insight for our purposes will be included, but for other results, we will merely include a citation to a work containing the proof. In addition, well-known or minor results, which will be used later, will be presented. All rings are assumed to be commutative with identity unless otherwise stated. Chapter 1 contains foundational results about the ring theory and graph theory. Chapter 2 contains some results about divisor graphs. Chapter 3 is about factorization as well as key examples and proofs.

Chapter 1

Introduction

1.1 Ring-theoretic definitions and elementary results

1.1.1 Rings and Ideals

This section will introduce definitions of various objects and terms referring to rings and ideals. These concepts will be helpful in the main results of this project.

Definition 1.1.1 ([8]). *A* **ring** *is a nonempty set R together with two binary operations $(+, \cdot)$ such that:*

a) $(R, +)$ is an abelian group.

b) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.

c) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

If $ab = ba$ for all $a, b \in R$, then we say that R is commutative. Additionally, if there is an element (denoted $1 = 1_R$) such that $1(a) = a(1) = a$ for all $a \in R$, R is said to have (multiplicative) identity. Unless otherwise specified, our rings will be commutative with identity.

Definition 1.1.2. *An element u of a ring R is called a* **unit** *if there exists an element v in R such that $uv = vu = 1$, where 1 is the multiplicative identity of the ring R. The set of units of R is denoted by $U(R)$.*

Definition 1.1.3 ([8]). *A nonempty subset $I \subseteq R$ is a* **ideal** *if and only if for all $a, b \in I$ and $r \in R$,*

a) $a - b \in I$,

b) $ra \in I$.

Definition 1.1.4 ([8])**.** *The* **ideal generated by the set X** *is the intersection of all ideals that contain this set:* $(X) = \bigcap_{X \subseteq I \subseteq R} I$. *Where the intersection ranges over all ideals containing* X.

We remark that $\langle X \rangle$ is equal to the set of all finite R-linear combinations of elements of X; that is,

$$(X) = \{\sum_{k=1}^{n} r_k x_k \mid r_k \in R, x_k \in X\}$$

.

The following definitions will introduce special types of ideals that will come in handy throughout the paper.

Definition 1.1.5. *Let* I *be an ideal in* R. *We say that* I *is a* **principal ideal** *if* I *is generated by a single element* $\alpha \in I$ *and we write* $I = (\alpha)$.

Definition 1.1.6. *Let* P *be an ideal in* R. *We say that* P *is a* **prime ideal** *if* $ab \in P$ *if* $a, b \in R$ *implies that either* $a \in P$ *or* $b \in P$.

Definition 1.1.7. *Let* M *be an ideal in* R. *We say that* M *is a* **maximal ideal** *if given any ideal* I *such that* $M \subseteq I$, *either* $I = M$ *or* $I = R$.

1.1.2 Polynomials

Here we recall the notion of a polynomial ring over a commutative ring with identity.

Definition 1.1.8. *The* **polynomial ring** $R[x]$ *in the indeterminate* x *with coefficients from* R *is the set of all formal sums*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

with addition given by

$$\sum a_i x^i + \sum b_i x^i = \sum (a_i + b_i) x^i$$

and multiplication given by

$$\left(\sum a_i x^i\right) \left(\sum b_i x^i\right) = \sum c_i x^i$$

where $c_i = \sum_{k=0}^{i} a_k b_{i-k}$.

If we have a fixed polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with $a_n \neq 0$, then we say that the degree of $f(x)$ is n and the leading coefficient of $f(x)$ is a_n.

Definition 1.1.9. *A polynomial $f(x)$ is called* **monic polynomial** *if its leading coefficient is equal to 1.*

1.1.3 Integral Domains and Fields

Here we introduce definitions of various objects and terms pertaining to integral domains and fields. Some of these concepts will be central to the main results.

Definition 1.1.10. *An element a in a ring R is called a* **zero divisor** *if there exists a nonzero element b in R such that $ab = 0$ or $ba = 0$.*

Proposition 1.1.1. *Let R be finite. Then $\alpha \in R$ is a zero divisor if and only if it is a nonunit.*

Proof. Assume that $\alpha \in R$ is a unit. Then, if $\beta \in R$ such that $\alpha\beta = 0$, note that $\beta = \alpha^{-1}\alpha\beta = 0$. Then if α is a unit, it is not a zero divisor. Notice that this portion of the proof does not depend on R being finite.

Now we can assume that α is not a zero divisor. Then for any $\beta, \gamma \in R$, such that if $\alpha\beta = \alpha\gamma$, then $\alpha(\beta - \gamma) = 0$. Since α is not a zero divisor, then $\beta - \gamma = 0$ implies that $\beta = \gamma$. In other words, the mapping $\alpha : R \to R$ such that $\alpha(\beta) = \alpha\beta$ is injective. Since R is finite, then this mapping must also be surjective, so there is some $\beta \in R$ such that $\alpha\beta = 1$. Thus α is a unit. $\square$

We remark that this result depends heavily on the fact that R is finite. For example, if $R = \mathbb{Z}$ then every element other than $-1, 0, 1$, is a nonunit, but none of them are zero divisors.

Definition 1.1.11. *A commutative ring with identity $1 \neq 0$ is called an* **integral domain** *if it has no nonzero zero divisors.*

Definition 1.1.12. *Let R be an integral domain.*

1. *Suppose $r \in R$ is nonzero and is not a unit. Then r is called* **irreducible** *in R if whenever $r = ab$ with $a, b \in R$, one of a or b must be a unit in R. Otherwise r is said to be reducible.*

2. *A nonzero element $p \in R$ is called* **prime** *in R if the ideal (p) generated by p is a prime ideal. in other words, a nonzero element p is a prime if it is not a unit and whenever $p \mid ab$ for any $a, b \in R$, then either $p \mid a$ or $p \mid b$.*

3. *Two elements a and b of R are said to be* **associates** *in R if $a = ub$ for some unit $u \in R$.*

If R is a domain, we denote the set of irreducibles of R by $Irr(R)$. Sometimes, for instance in the case when studying factorization in integral domains, we will generally not distinguish between an irreducible and its associates. So, we define $\overline{Irr}(R)$ to be a set of representatives, one from each equivalence class in the collection $\{\pi U(R) : \pi \in Irr(R)\}$.

Definition 1.1.13 ([8]). *A* **field** *F is a set together with two binary operations $+$ and $\cdot$ on F such that $(F, +)$ is an abelian group (call its identity 0) and $(F \setminus \{0\}, \cdot)$ is an abelian group, and the following distributive law holds:*

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c),$$

for all $a, b, c \in F$.

The following proposition is equivalent to the above definition of a field.

Proposition 1.1.2 ([8]). *A field is a commutative ring with identity such that (0) is a maximal ideal.*

We note here that every field is an integral domain. Furthermore, rings of polynomials are integral domains if the coefficients come from an integral domain. For example, the ring $\mathbb{Z}[x]$ of all polynomials in one variable with integer coefficients is an integral domain (this follows easily by a degree argument).

Proposition 1.1.3. *Any finite integral domain is a field.*

Proof. This is almost immediate from Proposition 1.1.1. Indeed since R is finite, each element is either a zero divisor or a unit. If the element is nonzero, it cannot be a zero divisor as R is a domain and hence must be a unit. $\square$

Definition 1.1.14. *A partially ordered set S satisfies the **Ascending Chain Condition (ACC)** if every ascending chain of elements in S stabilizes, i.e., if there is no infinite strictly increasing sequence*

$$x_1 < x_2 < x_3 < \cdots$$

where $x_i \in S$ for all i.

Definition 1.1.15. *A ring R is called Noetherian if it does not contain an infinite ascending chain of ideals. In other words, R is Noetherian if every ascending chain of ideals in R stabilizes.*

Definition 1.1.16. *The* **field of fractions** *of an integral domain R, denoted Frac(R), is the smallest field containing R as a subring.*

Definition 1.1.17. *Let $R \subseteq T$ be domains and $\pi \in T$. We say that π is **integral** over R if π is the root of a monic polynomial in $R[x]$.*

Definition 1.1.18. *A commutative ring R is said to be **integrally closed** if any element $a \in$ Frac(R) that is integral over R is actually in R.*

Definition 1.1.19. *A commutative ring R is called a* **Dedekind domain** *if it has the following three properties:*

1. *R is Noetherian,*
2. *R is integrally closed in its field of fractions K,*
3. *All nonzero prime ideals are maximal.*

1.1.4 Principal Ideal Domains

Now, we will investigate the properties of principal ideal domains.

Definition 1.1.20. *A* **principal ideal domain** *(PID) is an integral domain in which every ideal is principal.*

Example 1.1.1. *Some examples of PIDs may include:*

a) The integers $\mathbb{Z}$ is a PID.

b) If F is a field, then $F[x]$ is a PID.

Proposition 1.1.4. *In an integral domain, a nonzero prime element is always irreducible.*

Proof. Suppose (p) is a nonzero prime ideal and $p = ab$. Then $ab = p \in (p)$, so by the definition of a prime ideal, one of a or b, is in (p). Without loss of generality, assume $a \in (p)$, then $a = pr$ for some r. This implies that $p = ab = prb$ so $rb = 1$ and b is a unit. This shows that p is irreducible. $\square$

It is not true in general that an irreducible element is necessarily prime.

Example 1.1.2. *Consider the element* 3 *in the quadratic integer ring* $R = \mathbb{Z}[\sqrt{-5}]$*. Using a standard norm argument, it can be shown that* 3 *is irreducible in* R*, but* 3 *is not prime since* $(2+\sqrt{-5})(2-\sqrt{-5}) = 3^2$ *is divisible by* 3*, but neither* $2+\sqrt{-5}$ *nor* $2-\sqrt{-5}$ *are divisible by* 3 *in* R*.*

It is worth noting that if R is a principal ideal domain, the notions of nonzero prime and irreducible are the same. We now briefly recall the definition of greatest common divisor (gcd).

Definition 1.1.21. *Let* R *be an integral domain. If* $a, b \in R$*, we say that the element* d *is the* **greatest common divisor** *of* a *and* b *if* $d \mid a$*,* $d \mid b$*, and if* d' *is any other common divisor of* a *and* b *then* $d' \mid d$*.*

In general, two elements $a, b \in R$ may not have a greatest common divisor. In the case in which they do, we often write $\gcd(a, b)$ for the greatest common divisor of a and b. In the case where R is a PID then any two elements $a, b \in R$ have a greatest common divisor (which is the generator of the ideal (a, b)). We record this and more in the following proposition.

Proposition 1.1.5 ([11])**.** *Every nonzero prime ideal in a principal ideal domain is a maximal ideal.*

Proof. This is clear if R is a field, and so we will assume it is not. In this case there exists (p), a nonzero prime ideal. Let $I = (m)$ be any ideal containing (p). We must show that $I = (p)$ or $I = R$. Now $p \in (m)$ so $p = rm$ for some $r \in R$. Since (p) is a prime ideal and $rm \in (p)$, either r or m must lie in (p). If $m \in (p)$ then $(p) = (m) = I$. If, on the other hand, $r \in (p)$ write $r = ps$. In this case $p = rm = psm$, so $sm = 1$ (since R is an integral domain) and m is a unit so, $I = R$. $\square$

Proposition 1.1.6 ([11])**.** *In a principal ideal domain, a nonzero element is a prime if and only if it is irreducible.*

Proof. We have previously shown that prime implies irreducible. Now we much show that if p is irreducible, then p is prime, i.e., the ideal (p) is a prime ideal. If M is any ideal containing (p) then as R is a PID, $M = (m)$ is a principal ideal. Since $p \in (m)$, $p = rm$ for some r. But p is irreducible so by definition either r or m is a unit. This means either $(p) = (m)$ or $(m) = R$, respectively. Thus, the only ideals containing (p) are (p) and R. Therefore, (p) is a maximal ideal. Since maximal ideals are prime, the proof is complete. $\square$

Definition 1.1.22. *An* **atomic domain** *or* **factorization domain** *is an integral domain in which every nonzero nonunit can be written in at least one way as a finite product of irreducible elements.*

Definition 1.1.23. *A* **monoid** *is a set M together with an associative binary operation*

$$* : M \times M \longrightarrow M$$

with an identity element $1_M \in M$. That is for any $a, b, c \in M$, we have:

Closure: $a * b \in M$;

Associativity: $a * (b * c) = (a * b) * c$;

Identity: *There exists an element $1_M \in M$ such that $1_M * a = a * 1_M = a$;*

In other words, a monoid is a semigroup with an identity element.

Definition 1.1.24. *Let $(S, \circ)$ and $(T, *)$ be monoids. Let $\phi : S \longrightarrow T$ be a mapping such that*

$$\forall a, b \in S : \phi(a \circ b) = \phi(a) * \phi(b).$$

Suppose further that ϕ preserves identities, i.e.: $\phi(e_S) = e_T$. Then $\phi : (S, \circ) \longrightarrow (T, *)$ is a monoid homomorphism.

Definition 1.1.25. *A submonoid of a monoid $(M, \circ)$ is a subset N of M that is closed under the monoid operation and contains the identity element e of M. Symbolically, N is a submonoid of M if $e \in N \subseteq M$, and $x \circ y \in N$ whenever $x, y \in N$. In this case, N is a monoid under the binary operation inherited from M.*

Definition 1.1.26. *Let $(S, \circ)$ be a monoid. and $\phi : S \longrightarrow S$ a (monoid) isomorphism. Then ϕ is a monoid automorphism.*

Definition 1.1.27 ([11])**.** *A* **unique factorization domain** *(UFD) is an integral domain R in which every nonzero element $r \in R$ which is not a unit has the following two properties:*

1. *r can be written as a finite product of irreducible p_i of R (not necessarily distinct): $r = p_1 p_2 \cdots p_n$ and*

2. *The decomposition in a) is unique up to associates and reordering: namely, if $r = q_1 q_2 \cdots q_m$ is another factorization of r into irreducibles, then $m = n$ and there is some renumbering of the factors so that p_i is associate to q_i for $i = 1, 2, \ldots, n$.*

Note that a field F is trivially a unique factorization domain since every nonzero element is a unit, so there are no elements for which Properties a) and b) must be verified.

Proposition 1.1.7 ([11])**.** *In a unique factorization domain, a nonzero element is a prime if and only if it is irreducible.*

Proof. Let R be a unique factorization domain. Since we have shown that nonzero primes of R are irreducible, it remains to prove that each irreducible element is a prime. Let p be irreducible in R and assume that $p \mid ab$ for some $a, b \in R$. Now we need to show that p divides either a or b. To say that p divides ab is to say $ab = pc$ for some $c \in R$. Writing a and b as a product of irreducibles, we see from this last equation and from the uniqueness of the decomposition into irreducibles or ab that the irreducible element p must be associate to one of the irreducibles occurring either in the factorization of a or in the factorization of b. We can assume that p is associate to one of the irreducibles in the factorization of a, i.e., that a can be written as a product $a = (up)p_2 \cdots p_n$ for u a unit and some (possibly empty set of) irreducibles $p_2, \cdots, p_n$. But then p divides a, since $a = pd$ with $d = up_2 \cdots p_n$, completing the proof. □

Proposition 1.1.8 ([11])**.** *Let R be a PID, then R is a UFD.*

Proof. We must first show that R is atomic, i.e., every nonzero, nonunit element factors into irreducibles. Then we can show that R is in fact a UFD.

Let $\alpha_1 \in R$ be an arbitrary nonzero, nonunit element of R. If α_1 is irreducible, it trivially factors into irreducibles. Otherwise, we can factor $\alpha_1 = \alpha_2 \beta_2$, where neither α_2 nor β_2 is a unit. If both α_2 and β_2 factor into irreducibles, then so does α_1. Without loss of generality, assume that α_2 does not factor into irreducibles. Then we can write $\alpha_2 = \alpha_3 \beta_3$, with neither factor a unit. Now, continue factoring in this way to produce a sequence $\alpha_1, \alpha_2, \cdots$ of elements in R such that $(\alpha_1) \subsetneq (\alpha_2) \subsetneq \ldots$, with these inclusions being strict.

Now consider $I = \bigcup_{i=1}^{\infty}(\alpha_i)$. Due to the inclusion relation, I is an ideal in R. Then, since R is a PID, $I = (\alpha)$ for some $\alpha \in R$. So, $\alpha \in (\alpha_n)$ for some $n \in \mathbb{N}$. Also, we know that since $\alpha_n \in (\alpha)$, this means that $(\alpha_n) = (\alpha)$. Thus, for any $m \geq n$,

$$(\alpha_n) \subseteq (\alpha_m) \subseteq (\alpha)$$

which implies that $(\alpha_n) = (\alpha_m)$, contradicting the strict inclusions. Thus, α_1 must factor into irreducibles, meaning that the sequence of strict ideal inclusions above is finite. Now we show that this factorization is unique, as in the definition of a UFD.

Let $\pi \in R$ be an irreducible. If I is an ideal containing (π), we have that $I = (\alpha)$ for some $\alpha \in R$ and thus $\alpha \mid \pi$. Then either α is a unit, which means that $I = R$, or α is an associate of π, in this case $I = (\pi)$. Then (π) is a maximal ideal, so we know that (π) is a prime ideal and π is a prime element.

Now consider two factorizations

$$\pi = \pi_1 \cdots \pi_m = \tau_1 \cdots \tau_n,$$

with each π_i, τ_i irreducible in R.

Without loss of generality, assume that $m \geq n$. Since π_1 is prime and $\pi_1 \mid \tau_1 \cdots \tau_n$, we have by induction that $\pi_1 \mid \tau_j$ for some j, $1 \leq j \leq n$. So, without loss of generality, assume that $\pi_1 \mid \tau_1$. Then $(\tau_1) \subseteq (\pi_1) \neq R$, so by maximality of (τ_1), we have $(\tau_1) = (\pi_1)$, that is, π_1 and τ_1 are associates. Thus, $\pi_1 = u_1\tau_1$ for some $u_1 \in U(R)$, so

$$u_1\pi_2 \cdots \pi_m = \tau_2 \cdots \tau_n.$$

Note that $u_1\pi_2$ is still irreducible. We can reorder, if necessary, and use induction to reduce this equality to $u\pi_{n+1} \cdots \pi_m = 1$, with $u \in U(R)$. However, since each π_i is irreducible (and thus a nonunit), we get a contradiction unless $m = n$. Thus, $m = n$ and each π_i is an associate of τ_i, meaning that R is in fact a UFD. $\square$

Definition 1.1.28. *Recall that R is a* **Finite Factorization Domain** (**FFD**) *if R is atomic and every nonzero nonunit element of R has finitely many non-associate irreducible divisors.*

Definition 1.1.29. *An integral domain D is called a* **Half Factorial Domain** (**HFD**) *if it satisfies the following conditions:*

1. *Every nonzero element of D that is not a unit can be factored into a product of a finite number of irreducibles;*

2. *If $p_1p_2\cdots p_m = q_1q_2\cdots q_n$ are factorizations a into irreducibles, then $m = n$.*

1.2 Graph-theoretic definitions

A graph $G = (V, E)$ is a set V, called the vertex set, and a set of irreflexive, symmetric relations E, on V, called the edge set. If x and y are distinct vertices of Γ, that is to say, $x, y \in V$ with $x \neq y$, then if x and y are related in E, we call the relation an edge between x and y, denoted by (x, y). Note that if (x, y) is an edge, then (y, x) denotes the same edge. We will usually demand that our graphs are simple in the sense that there is, at most, one edge between two fixed vertices and no loops (an edge that begins and ends at the same vertex).
It is necessary to introduce some key graph-theoretic definitions.

Definition 1.2.1. *The* **degree**$ofavertexisthenumberofedgesincidenttoit.$

Definition 1.2.2. *A* **subgraph** *G' of a graph G is a graph G' whose vertex set and edge set are subsets of those of G (and if the edge e is in G' then both vertices to which e is incident are in G').*

Definition 1.2.3. *A* **complete graph** *is a graph in which each pair of graph vertices is connected by an edge. The complete graph with n graph vertices is denoted K_n.*

If (x, y) is an edge, we say that x and y are adjacent, and when convenient we will denote it by $x - y$. A path of length n from a vertex x to a distinct vertex y is a sequence of $n + 1$ distinct vertices $x = V_0, V_1, \ldots, V_n = y$ such that V_i and V_{i+1} are adjacent for $0 \leq i \leq n - 1$. For convenience, we will usually denote such a path by $V_0 - V_1 - \cdots - V_n$. If x and y are vertices of a graph, we define the distance between x and y, $d(x, y)$, to be the length of the shortest path between them. If no path exists between x and y, we say that $d(x, y) = \infty$. If in a graph Γ there are vertices x and y such that $d(x, y) = \infty$, we say that the graph is disconnected. We define a cycle by requiring that $x = y$ (and $n \geq 3$) in the above definition of a path. Note that for both the path and the cycle, the length is just the number of edges determined by the $\{v_i \mid 0 \leq i \leq n\}$. The diameter of a graph G, denoted $diam(G)$, is defined to be the maximum of the distances $d(x, y)$ as x and y vary over all vertices in the graph. The girth of a graph is the length of the shortest cycle.

Definition 1.2.4. *A* **cut vertex** *in the graph G is a vertex that when removed (along with all of its incident edges), the corresponding graph has more connected components than G.*

Definition 1.2.5. *We say that a vertex of the graph G is a* **leaf** (**or end**) *if this vertex is connected to exactly one other vertex in the graph.*

Definition 1.2.6. *A graph G is* **connected** *if and only if every pair of vertices in G is connected.*

Definition 1.2.7. *Two graphs G and H are* **isomorphic** *if there exists a bijection $f : V(G) \to V(H)$ such that $(u, v) \in E(G)$ if and only if $(f(u), f(v)) \in E(H)$ for all $u, v \in V(G)$.*

Chapter 2

Divisor Graph

2.1 The Divisor Graph

This section will cover some main results of this project. Also, we explore key examples that provide useful insight to help us understand the main results more clearly.

Let R be an integral domain and denote the set of nonzero nonunits of R by $R^\bullet$. If x is a nonzero nonunit, we define the set $D_x = \{d \in R^\bullet | \frac{x}{d} \in R^\bullet\}$ (intuitively, this is the set of proper nonunit divisors of x in R). On the set D_x, we impose the equivalence relation $\sim$ by declaring that for $a, b \in D_x$, $a \sim b$ if and only if $a = ub$ for some unit $u \in R$. This set of equivalence classes shall be written as $\overline{D_x}$. We now associate a graph with this collection of equivalence classes.

Definition 2.1.1. *We define the divisor graph of the element $x \in R^\bullet$ denoted as $G(x) = (V, E)$, by declaring that $V = \overline{D_x}$ and that $(a, b) \in E$ if and only if ab divides x in R.*

This graph is in the spirit of the irreducible divisor graph defined in [9], and in fact, the irreducible divisor graph is almost a subgraph of the divisor graph. The distinction is that the irreducible divisor graph does allow for loops.

Definition 2.1.2 ([9])**.** *Let R be a domain, and let $x \in R$ be a nonzero nonunit that can be factored into irreducibles. We define the irreducible divisor graph of x to be the graph $G(x) = (V, E)$ where $V = \{y \in \overline{Irr}(R) : y \mid x\}$, and for given $y_1, y_2 \in \overline{Irr}(R)$ we have the edge $(y_1, y_2) \in E$ if and only if $y_1 y_2 \mid x$. Furthermore, we attach $n-1$ loops to the vertex y if y^n divides x.*

We now recall the notion of the classical zero divisor graph of Anderson and Livingston. We highlight this to point out that the divisor graph, generalized to rings that are not necessarily domains, also has an intimate connection with the zero divisor graph. Indeed, $\overline{D_x}$ naturally coincides with the zero divisor graph if the verticies are restricted to the nonzero zero divisors.

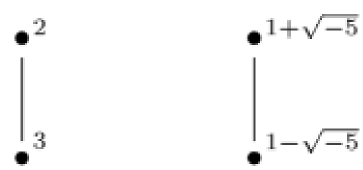
Definition 2.1.3. *Let R be a commutative ring with a nonzero identity. We define the zero divisor graph of R, denoted $\Gamma(R)$, to be a simple graph with vertex set being the set of nonzero zero divisors of R and with (x, y) an edge if and only if $x \neq y$ and $xy = 0$.*

The following three examples are from [9].

Example 2.1.1. *Let $R = \mathbb{Z}[\sqrt{-5}]$. Using norms, it is easy to see that the only non associate irreducible factorizations of 6 are:*

$$6 = (2)(3) = (1 - \sqrt{-5})(1 + \sqrt{-5})$$

So, G(6) is as follows:

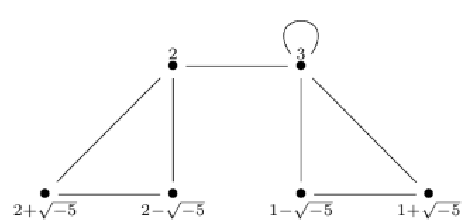


In this case the irreducible divisor graph coincides with the divisor graph.

Example 2.1.2. *As another example, the only irreducible factorizations of 18 in R are:*

$$18 = (2)(3^2) = 3(1 - \sqrt{-5})(1 + \sqrt{-5}) = 2(2 + \sqrt{-5})(2 - \sqrt{-5})$$

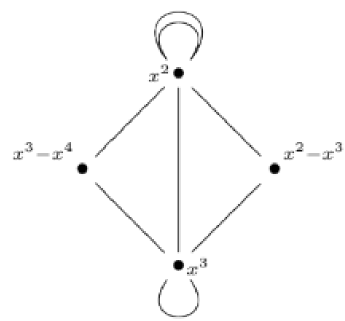
So, G(18) is as follows:



Example 2.1.3. *Let $R = \mathbb{Q}[x^2, x^3]$ and let $f(x) = x^9 - x^{10}$. The only irreducible factorizations of f in R are:*

$$f(x) = x^2 \cdot x^2 \cdot x^3(x^2 - x^3) = x^2 \cdot x^2 \cdot x^2(x^3 - x^4) = x^3 \cdot x^3(x^3 - x^4)$$

And $G(f)$ is as follows:



Proposition 2.1.1 ([9])**.** *Let R be an atomic domain. Then R is an FFD if and only if $G(x)$ is finite for each nonzero nonunit $x \in R$.*

Proof. If R is an FFD, then every nonzero nonunit $x \in R$ has only finitely many non-associate irreducible factors, whence the vertex set of $G(x)$ is finite. On the other hand, if $G(x)$ is finite for each x, then each x has only finitely many nonassociate irreducible divisors, and R is an FFD. □

How close is the graph to its subgraph, when can we be assured that they are the same? Any K_n can be realized as an irreducible divisor graph.

Theorem 2.1.1. *Let $x \in R^\bullet$. If x is irreducible then $G(x)$ consists only of the vertex x. If x is not irreducible and x has a prime divisor, then $G(x)$ is connected and $diam(G(x)) \leq 2$.*

Example 2.1.4. *Let R be a Dedekind domain with $Cl(R) \cong \mathbb{Z}/2\mathbb{Z}$ and $\mathfrak{P}$ a nonprincipal prime ideal. Then $\mathfrak{P}^2 = (\alpha)$ and the powers of α all factor uniquely (in this case, α is the only irreducible dividing α^n for all n as $(\alpha^n) = \mathfrak{P}^{2n}$). It is easy to see that $G(\alpha^n)$ is a single vertex with $n-1$ loops.*

Theorem 2.1.2. *Let $x \in R$ be a nonzero nonunit that has a factorization of length 2. Then $G(x)$ is connected if and only if x factors uniquely.*

Proof. If x factors uniquely it is obvious that $G(x)$ is connected, so we will prove the forward direction.

Suppose that $x = \pi_1\pi_2$ and $G(x)$ is connected. If x has another irreducible factorization $x = \xi_1\xi_2\cdots\xi_n$ then the fact that $G(x)$ is connected shows that (without loss of generality) π_1 is adjacent to ξ_1. So there is a factorization of x of the form $x = \pi_1\xi_1\eta$. But comparing this with the factorization $x = \pi_1\pi_2$, we obtain that η must be a unit and ξ_1 is associated to π_2. With this knowledge, we compare again the factorizations $x = \pi_1\pi_2 = \xi_1\xi_2\cdots\xi_n$ and obtain that $\pi_1 = \xi_2\cdots\xi_n$ and so $n = 2$ and the factorization of x is unique. $\square$

Chapter 3

Factorization and factorization type

3.1 Factorization

Factorization, the study of how elements factor multiplicatively (or sometimes additively in the case of monoids) is a central topic in commutative algebra. In this section we will explore the factorization properties of a fixed element in a monoid (or sometimes a domain).

Definition 3.1.1. *Let M and M' be monoids and $x \in M$ and $y \in M'$ both be atomic elements. For any $x \in M$, we define M_x to be the submonoid of M generated by Irr(x) (the irreducible divisors of x).*

Proposition 3.1.1. *Suppose that $f : M_x \longrightarrow M'_y$ is a bijection satisfying $f(ab) = f(a)f(b)$. Then $\pi \in M_x$ is irreducible if and only if $f(\pi) \in M'_y$ is irreducible.*

Proof. Suppose that $\pi \in M_x \subseteq M$ is an irreducible and consider $f(\pi) \in M'_y$. Let $f(\pi) = \alpha\beta$ with α, β both nonunits in $M'_y \subseteq M'$. Since f is surjective, we can find $a, b \in M_x$ (necessarily nonunits) such that $\alpha = f(a)$ and $\beta = f(b)$. By injectivity of f, this gives $\pi = ab$. The proof that $f(\pi)$ is irrededucible implies that π is irreducible is almost identical. □

We now formally define what it means for two factorizations to be "the same." This is the obvious analog of the condition of unique factorization in a domain (or monoid).

Definition 3.1.2. *Let $x \in M$ be a nonunit and suppose that x has the irreducible factorizations*

$$x = \pi_1 \pi_2 \cdots \pi_n = \xi_1 \xi_2 \cdots \xi_m$$

with each $\pi_i, \xi_j \in M_x$. We say that these two factorizations are equivalent if

1. *$n = m$ and*

2. *there is a $\sigma \in S_n$ such that for all $1 \leq i \leq n$, $\pi_i = u_i \xi_{\sigma(i)}$ with $u_i \in U(M)$.*

The set of all factorizations of x are partitioned into equivalence classes by the above notion of equivalence. Although there may be many different factorizations in a single equivalence class (by introducing units and/or permuting the order), we will often think of an equivalence class as a single factorization.

Example 3.1.1. *Consider a monoid with elements x and y which factor as $x = abc = cde = efg = ghi$ and $y = a'b'c' = c'd'e' = b'f'g' = a'h'i'$ with all elements (except x and y) irreducible. The irreducible divisor graphs are not isomorphic (in the second all three degree 4 vertices are adjacent, but this is not the case for the first one). In fact, for these elements x and y the irreducible divisor graphs are "close" (same number of vertices of each degree, same number of 3-cycles, both connected and non-bipartite, both planar) but are not isomorphic. So x and y, although they have the same number of irreducible divisors and both have precisely 4 factorizations, all square-free, of length 3, should be of different factorization type. Additionally, for both sets of factorizations, comparing the pairs, we see that three of the six pairs have precisely one irreducible in common. The key difference seems to be that for y the factorization abc has common irreducibles with all other and for x this is not true.*

Definition 3.1.3. *Let M be a monoid and $x \in M$ we define $div(x)$ to be the collection of equivalence classes of divisors of x up to associates.*

we will, for simplicity, consider our monoids to be *reduced* in the sense that $1 \in M$ is the only unit. This reduction is common in the theory of monoid factorization, as it is easy to pass from a monoid to its reduction without any loss of significant factorization information.

Definition 3.1.4 (*Factorization type*)**.** *We say that $x \in M$ and $y \in M'$ have the same factorization type if there is a bijection $f : div(x) \longrightarrow div(y)$ satisfying*

1. $f(x) = y$

2. $f(1_M) = 1_{M'}$

3. For all $a_1, a_2 \in$ *div*(x),

  (a) if $a_1 a_2 \in$ *div*(x) *then* $f(a_1a_2) = f(a_1)f(a_2)$ *and*

  (b) $a_1a_2 \in$ *div*(x) *if and only if* $f(a_1)f(a_2) \in$ *div*(y).

Lemma 3.1.1. *If x and y have the same factorization type and $f : div(x) \longrightarrow div(y)$ is our bijection then $f|_{Irr(x)} : Irr(x) \longrightarrow Irr(y)$ is a bijection.*

Proof. Suppose that $a \in \mathrm{div}(x)$ is irreducible and that $f(a) = y_1y_2$ where $y_1, y_2 \in \mathrm{div}(y)$. Since f is onto, there are $b_1, b_2 \in \mathrm{div}(x)$ such that $f(b_1) = y_1, f(b_2) = y_2$. So we have $f(a) = f(b_1)f(b_2) = f(b_1b_2)$ and so $a = b_1b_2$, contradicting the irreducibility of a. Hence $f|_{\mathrm{Irr}(x)} : \mathrm{Irr}(x) \longrightarrow \mathrm{Irr}(y)$. It remains to establish that $f|_{\mathrm{Irr}(x)}$ is a surjection. So let $\xi \in \mathrm{Irr}(y)$, and select $\pi \in \mathrm{div}(x)$ such that $f(\pi) = \xi$. If $\pi = a_1a_2$ then $\xi = f(a_1)f(a_2)$ and the proof is complete. □

Example 3.1.2. *To see why condition 3(b) is needed, consider the ring $F[x^4, x^5]$. In this case $div(x^9) = \{1, x^4, x^5, x^9\}$ and $div(x^{12}) = \{1, x^4, x^8, x^{12}\}$. If we define $f : div(x^9) \longrightarrow div(x^{12})$ by $f(x^4) = x^4$ and $f(x^5) = x^8$ then this satisfies the assumptions of the definition of same factorization type with the exception of 3(b). Note that $f(x^4)f(x^4) = x^8 \in div(x^{12})$, but $x^4x^4 \notin div(x^9)$. Also note in this example that the function f does not preserve the irreducibles. In fact, $div(x^9)$ has two irreducibles and $div(x^{12})$ has only one.*

Definition 3.1.5. *Let M be an atomic monoid and $x \in M$. We define the elasticity of x as follows.*

$$\rho(x) = sup(\{\tfrac{n}{m} \mid x = \alpha_1 \cdots \alpha_n = \beta_1 \cdots \beta_m; \alpha_i, \beta_j \in Irr(M)\}).$$

The elasticity of a monoid (or domain) is taken as the supremum over the elasticity of the elements contained within the monoid (or domain).

Proposition 3.1.2. *If x and y have the same factorization type, then $\rho(x) = \rho(y)$.*

Proof. Let $x = \pi_1\pi_2 \cdots \pi_n$ be an irreducible factorization. Then $y = f(x) = f(\pi_1)f(\pi_2)\cdots f(\pi_n)$ and by Lemma 3.1.1 each $f(\pi_i)$ is irreducible. So y has a corresponding factorization of length n. Now suppose that $y = \xi_1\xi_2 \cdots \xi_m = f(a_1)f(a_2)\cdots f(a_m) = f(a_1a_2\cdots a_m) = f(x)$ and so $a_1a_2\cdots a_m = x$, and the proof that each a_i is irreducible is similar to the above. Hence each factorization of x corresponds to an equal length factorization of y and so $\rho(x) = \rho(y)$. $\square$

The next result also underscores the importance of condition $3(b)$ in the definition of factorization type as it is crucial for assuring that the notion of factorization type is symmetric.

Theorem 3.1.1. *Let M be a monoid and $a, b \in M$. The relation given by $a \sim b$ if a and b have the same factorization type is an equivalence relation.*

Proof. $\sim$ is clearly reflexive. If we have $a \sim b$, then the bijection $f^{-1} : \mathrm{div}(b) \longrightarrow \mathrm{div}(a)$ satisfies $f^{-1}(b) = a$ and $f^{-1}(1) = 1$. So suppose that $b_1, b_2 \in \mathrm{div}(b)$ (with $a_i = f^{-1}(b_i)$). If $b_1b_2 = f(a_1a_2) \in \mathrm{div}(b)$ then $a_1a_2 = f^{-1}(b_1b_2) = f^{-1}(b_1)f^{-1}(b_2)$. We also note that $b_1b_2 \in \mathrm{div}(b)$ if and only if $a_1a_2 \in \mathrm{div}(a)$ and so the relation is symmetric.

For transitivity, suppose that $f : \mathrm{div}(x) \longrightarrow \mathrm{div}(y)$ and $g : \mathrm{div}(y) \longrightarrow \mathrm{div}(z)$. If $a_1, a_2 \in \mathrm{div}(x)$ then $f(a_1), f(a_2), f(a_1)f(a_2)$, and $f(a_1a_2)$ are all in $\mathrm{div}(y)$. Hence $g(f(a_1)f(a_2)) = gf(a_1)gf(a_2)$ and $a_1a_2 \in \mathrm{div}(x)$ if and only if $gf(a_1)gf(a_2)$ are in $\mathrm{div}(z)$. $\square$

Lemma 3.1.2. *If $a_1, a_2 \in div(x)$ then $a_1 \mid a_2$ if and only if $f(a_1) \mid f(a_2)$.*

Proof. If $a_2 = za_1$ then $f(a_2) = f(z)f(a_1)$ and hence $f(a_1)$ divides $f(a_2)$. On the other hand, if $f(a_1)$ divides $f(a_2)$ then there is a $z \in \mathrm{div}(y)$ such that $f(a_2) = zf(a_1)$. Since f is surjective, there is a $z' \in \mathrm{div}(x)$ such that $z = f(z')$. So $f(a_2) = f(z')f(a_1)$. By 3(b), $z'a_1 \in \mathrm{div}(x)$ and so by 3(a), $f(a_2) = f(z'a_1)$ and since f is one-to-one, $a_2 = z'a_1$ and hence $a_1 \mid a_2$. $\square$

Theorem 3.1.2. *If x and y have the same factorization type, then $G(x) \cong_G G(y)$ ($G(x)$ is the irreducible divisor graph).*

Proof. Suppose that $x \in M$ and $y \in M'$ have the same factorization type. We use the bijection f. Note that by Lemma 3.1.1 this give a bijection between the vertex sets of $G(x)$ and $G(y)$. We merely need to show that for $a_1, a_2 \in \mathrm{div}(x)$ that a_1 and a_2 are adjacent if and only if $f(a_1)$ and $f(a_2)$ are adjacent.

Suppose first that a_1 and a_2 are adjacent in $G(x)$. Then $a_1a_2 \in \mathrm{div}(x)$ and so $f(a_1a_2) = f(a_1)f(a_2)$ and hence $f(a_1)$ and $f(a_2)$ are adjacent in $G(y)$.

Now suppose that $f(a_1)$ and $f(a_2)$ are adjacent in $G(y)$. Then $f(a_1)f(a_2) \in \mathrm{div}(y)$, hence $a_1a_2 \in \mathrm{div}(x)$ and so a_1 and a_2 are adjacent in $G(x)$. $\square$

Example 3.1.3. *If we have the element z with factorizations $z = abx_1 = bcx_2 = cdx_3 = dex_4 = efx_5 = afx_6$ then $G(z)$ has a 6-cycle and has diameter 4. This example can be adjusted to show that $G(x)$ can have arbitrarily large diameter. The girth of any of these appears to be either 3 or ∞.*

Example 3.1.4. *Consider the elements $x = a^2bc = adef$ and $y = abc = a^2def$. These two elements have different factorization type (for example, x has 7 subfactorizations of length 3 and y has 8 subfactorizations of length 3). But $G(x) \cong_G G(y)$.*

Example 3.1.5. *Consider these factorizations*

$$z = abx_1x_2x_3y_1 = acx_4x_5x_6y_2 = adx_7x_8x_9y_3$$
$$= bcx_{10}x_{11}x_{12}y_4 = bdx_{13}x_{14}x_{15}y_5 = cdx_{16}x_{17}x_{18}y_6$$

and

$$t = abx_1x_2x_3y_1 = acx_4x_5x_6y_2 = adx_7x_8x_9y_3 = bcx_{10}x_{11}x_{12}y_4$$
$$= bdx_{13}x_{14}x_{15}y_5 = cdx_{16}x_{17}x_{18}y_6 = abcd$$

It is easy to see that both z and t have the same irreducible divisor graph but their elasticity are different, $\rho(z) = 1$ and $\rho(t) = \frac{6}{4}$.

Example 3.1.6. *Let $x = abx_1y_1 = acx_2y_2 = adx_3y_3 = bcx_4y_4 = bdx_5y_5 = cdx_6y_6$ and $y = abx_1y_1 = acx_2y_2 = adx_3y_3 = bcx_4y_4 = bdx_5y_5 = cdx_6y_6 = abcd$. $\rho(x) = \rho(y) = 1$ and $G(x)$ and $G(y)$ coincide. But there are factorizations with more than 3 irreducible factors and G contains a K_4.*

Definition 3.1.6. *Let G be a graph and $K_m \subseteq G$. For this K_m we let n be the maximal integer such that $K_m \subseteq K_n \subseteq G$. We call this K_n a maximal complete subgraph of G.*

Definition 3.1.7. *Let G be a graph. We say that a $K_n \subseteq G$ that does not correspond to a factorization is a "false K_n".*

Example 3.1.7. *If we consider $z = a_1a_2x_1 = a_1a_3x_2 = a_1a_4x_3 = a_2a_3x_4 = a_2a_4x_5 = a_3a_4x_6$, then this set of irreducible factorizations gives a "false K_4" in the graph. Note that each 3−cycle* *corresponding to an irreducible factorization is a maximal K_3. There appear to be some limitations on how these can be built, but it appears that false K_n can be made arbitrarily long.*

Theorem 3.1.3. *In the irreducible divisor graph of x, where x is square free, the following conditions are equivalent.*

1. *The irreducible divisor π is a leaf of $G(x)$.*

2. *π is a vertex in a connected component of $G(x)$ that is isomorphic to K_2.*

Proof. (2) implies (1) is straightforward. For (1) implies (2), suppose that π is adjacent to the irreducible element ξ and suppose that ξ is adjacent to η. Then $\pi\xi a_1 \cdots a_n = \xi\eta b_1 \cdots b_m$. Hence $\pi a_1 \cdots a_n = \eta b_1 \cdots b_m$. Since π is only adjacent to ξ and x is square free, $a_1 \cdots a_n = 1$. And so η divides π which is the desired contradiction. □

3.2 Divisors and Factorization

In this section, we will talk about both monoids and domains (the monoid structure will be written multiplicatively).

There has been much work done with regard to factorization in monoids and domains. The simplest structures with regard to factorization are UFMs/UFDs where the factorizations are unique. In more general situations, factorizations into irreducibles may not be unique (or even exist in some cases).

It is well known that any factorization into primes is necessarily unique. There are cases in which factorization into nonprime irreducibles is unique, but this is impossible globally. We will make this precise.

Theorem 3.2.1. *If R is an atomic domain, then every irreducible factorization is unique if and only if every irreducible element is prime.*

It is worth noting that in the nonatomic situation, the previous result may not hold.

Theorem 3.2.2. ([*10*]) *There exists a nonatomic domain with a unique (up to associates) irreducible.*

Note that in such a domain, every element that *can* be factored into irreducibles has a unique atomic factorization, for if π is the irreducible of the previous theorem, then every atomic factorization is (up to associates) of the form π^n for some $n \in \mathbb{N}$.

We now produce a number of motivating examples.

Example 3.2.1. *Consider first the domain $\mathbb{Q}[x^2, x^3]$. In this domain, the element x^4 factors (uniquely) into (nonprime) irreducibles via $x^4 = x^2.x^2$ and the element $(1+x^2)^2$ is the square of a prime. Both of these elements are the square of a single irreducible, but since only one of them is prime, these should "philosophically" be different in some sense (even though the factorizations of the elements themselves do not seem to distinguish from the information given alone).*

Example 3.2.2. *Consider the ring $R := \mathbb{Z}[\sqrt{-38}]$ of algebraic integers with class group isomorphic to $\mathbb{Z}_6$. If P_k is a prime ideal in the ideal class corresponding to $\overline{k} \in \mathbb{Z}_6$, then we consider the elements $a, b \in R$ where $(a) = (P_1P_5)(P_3P_3)$ and $(b) = (P_2P_4)(P_3P_3)$. The elements (principal ideals) P_1P_5, P_2P_4 and P_3P_3 are all irreducible and a quick examination of the relations in the class group demonstrate that both a and b factor uniquely.*

As before, these factorizations look the same in a certain sense, but as P_1P_5 is a product of primes of order 6 in the class group and P_2P_4 is a product of primes of order 3, somehow these should be different because we have

$$\alpha^3 = (P_1P_5)(P_1P_5)(P_1P_5) = (P_1P_1P_1)(P_5P_5P_5)$$

and

$$\beta^3 = (P_2P_4)(P_2P_4)(P_2P_4) = (P_2P_2P_2)(P_4P_4P_4)$$

so α^3 still factors uniquely, but $\beta^3 = yz$ where $y = P_2^3$ and $z = P_4^3$

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