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Cohen-Macaulay Type of Open Neighborhood Ideals of Unmixed Trees

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Cohen-Macaulay Type of Open Neighborhood Ideals of Unmixed Trees

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Abstract

Given a tree T and a field k, we define the open neighborhood ideal $N(T)$ of T in $k[V]$ to be the ideal generated by the open neighborhoods of all vertices in the graph. If T is unmixed with respect to the total domination problem, then it is known that $N(T)$ is Cohen-Macaulay. Our goal is to compute the (Cohen-Macaulay) type of $k[V]/N(T)$ using graph theoretical properties of T. We achieve this by using homological algebra and properties of monomial ideals. Along the way, we also provide a different characterization of unmixed trees and a generalization of the total dominating problem with the corresponding decomposition theorem.

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Chapter 1

Introduction

Combinatorial Commutative Algebra is a branch of mathematics that uses combinatorics and graph theory to understand algebraic constructions. Consider a polynomial ring $R = k[X_1, \ldots, X_d]$ over some field k with d variables. Given an ideal $I \leq R$ whose generators are monomials, we want to understand properties of the quotient ring R/I. Specifically, we can ask questions like "how can I be written as an intersection of irreducible ideals in R just like the prime factorization of integers," and "how do various types of algebraic invariants change when we modify I." One way to study these ideals is by realizing a combinatorial structure that hides in R and R/I. It is not hard to see that the set of all monomials in R with a partial ordering defined by division (for monomials f and g in R, $f < g$ if and only if f divides g) forms a poset; let's denote this poset by $(P, <)$. Hence the monomials in the ideal I can be viewed as a subposet $(P_I, <)$ of $(P, <)$, whose minimal elements are the minimal monomial generators of I. In particular, the monomials in R/I correspond to the monomials in P that are not in P_I.

Much is known when the monomial generators of I are squarefree, using simplicial complexes. A *simplicial complex* on the set $V := \{X_1, \ldots, X_d\}$ is a nonempty collection of subsets of V that is closed under the subset relation $\subseteq$. Treating each squarefree monomial as a subset of V (for instance, the monomial $X_1X_2X_4$ corresponds to the set $\{X_1, X_2, X_4\}$), the set $P \setminus P_I$ restricted to the squarefree monomials gives a simplicial complex. In fact, there is a one-to-one correspondence between the squarefree monomial ideals of R and the simplicial complexes on V. This intuition gave rise to Stanley-Reisner theory and many properties of R/I have been studied this way. For instance, the irreducible decomposition of I can be explicitly given in terms of the facets (i.e., the maximal elements) of the simplicial complex.

In this thesis, we study squarefree monomial ideals from a slightly different approach. Instead of simplicial complexes, we use finite simple graphs. Given a graph $G = (V, E)$ with the vertex set V defined as before, we define the open neighborhood ideal of G denoted $N(G)$ to be the ideal generated by the open neighborhoods of each vertex in G. The ideal $N(G)$ is a squarefree monomial ideal of R. The decomposition of $N(G)$ can be computed directly from G by solving the total domination problem for G [7, Theorem 4.1.5]. Also, in the case when every minimal solution for the total dominating problem has the same size (such graphs are called domination-unmixed), if G is a tree, then $N(G)$ is Cohen-Macaulay [7, Theorem 4.3.14].

The goal of this thesis is to compute one of the algebraic invariants called the (Cohen-Macaulay) type of the quotient $R/N(G)$ if G is an unmixed tree. The type of $N(G)$ is a measure of the complexity of $N(G)$. More specifically, it measures the number of ideals in the irreducible decomposition of the image of $N(G)$ in an appropriate quotient ring of R. We achieve our goal in Theorem 3.3.9 below, where we show that the computation of this algebraic invariant only requires the graph theoretic properties of G. In Chapter 2, we provide some definitions and preliminary results from homological algebra, commutative algebra, and graph theory that are used in the proof of our main result.

Chapter 2

Main Definitions and Background

In this chapter, we give the necessary background to prove the main results in Chapter 3. Most of the definitions and facts in the following sections are from [10], [11], [12], [13], and [14].

Conventions throughout this thesis. Let R be a commutative ring with identity, and let M be an R-module. Let $I \subseteq R$ be an ideal; signified as $I \leq R$. Assume that $\mathbb{N} := \{1, 2, 3, \dots\}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}$, set $[n] := \{1, 2, \dots, n\}$. Given $S \subseteq R$, we denote the ideal generated by S in R by $(S)R$, or simply $\langle S \rangle$ if the ring is clear from context. If the elements are given explicitly, say $f_1, \dots, f_n$, we write $(f_1, \dots, f_n)R$ instead of $(\{f_1, \dots, f_n\})R$. Similarly, we write $\langle f_1, \dots, f_n \rangle$ instead of $\langle \{f_1, \ \dots \ , f_n\} \rangle$. For a given set S, we denote the power set of S by 2^S. The ideal generated by the empty set is just the zero ideal; i.e., $\langle \emptyset \rangle = 0$. Given $I \leq R$ and $f \in R$, we sometimes use the bar notation $\overline{f} := f + I \in R/I$.

2.1 Localization and Regular Sequences

In this section, we briefly introduce 'localization' and 'regular sequence.'

Definition 2.1.1. We say R is *local* if R has a unique maximal ideal $\mathfrak{m}$. The *residue field* of R is $R/\mathfrak{m}$. The tuples $(R, \mathfrak{m}, k)$ or $(R, \mathfrak{m})$ denote the local ring R with maximal ideal $\mathfrak{m}$ and $k := R/\mathfrak{m}$.

Given an arbitrary ring R that is not local, we can use 'localization,' defined next, to construct a local ring.

Definition 2.1.2. Let $U \subseteq R$ be a multiplicatively closed set, meaning U is closed under multiplication and contains 1. Set

$$U^{-1}M := \{\text{equivalence classes of } \sim\}$$

where $\sim$ is the equivalence relation on $M \times U$ defined by $(m, u) \sim (n, v)$ if there exists $w \in U$ such that $w(vm - un) = 0$. We denote the equivalence class of (m, u) as $\frac{m}{u}$ or m/u. The *localization* of M at U is an R-module $U^{-1}M$ with addition and scalar multiplication defined by

$$\frac{m}{u} + \frac{n}{v} := \frac{vm + un}{uv} \quad \text{and} \quad r \cdot \frac{m}{u} := \frac{rm}{u}.$$

for all $r \in R,\ m, n \in M,\ u, v \in U$.

Since R itself is an R-module, we can localize R at U in the same way, equipped with the multiplication defined by

$$\frac{r}{d} \cdot \frac{r'}{d'} := \frac{rr'}{dd'}$$

for all $r, r' \in R,\ d, d' \in U$. It is straightforward to show that $U^{-1}R$ is also a ring and

$U^{-1}M$ is a $U^{-1}R$-module via the multiplication

$$\frac{r}{u} \cdot \frac{m}{v} = \frac{rm}{uv}.$$

Fact 2.1.3. For any prime ideal $\mathfrak{p} \leq R$, the set $R \setminus \mathfrak{p}$ is multiplicatively closed.

Notation 2.1.4. Set $M_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}M$ for any prime ideal $\mathfrak{p} \leq R$. Also, for an ideal $I \leq R$, we denote

$$IM := \left\{ \sum_{i=1}^{n} r_i m_i : n \in \mathbb{N}_0, \forall i \in [n], r_i \in I \text{ and } m_i \in M \right\}.$$

If $I = \langle r_1, \ldots, r_n \rangle$, then write $(r_1, \ldots, r_n)M := IM$.

Fact 2.1.5 ([3, Proposition 15.38])**.** Let U be a multiplicatively closed set of R. Then there is a bijection f

$$f : \{I : I \leq R,\ I \cap U = \emptyset\} \to \{I' : I' \leq U^{-1}R\}$$

given by

$$f(I) = U^{-1}I := \left\{ \frac{a}{b} : a \in I,\ b \in U \right\}.$$

By Fact 2.1.5 and Fact 2.1.3, for any prime ideal $\mathfrak{p} \leq R$, the localization $R_{\mathfrak{p}}$ has a unique maximal ideal $\mathfrak{p}_{\mathfrak{p}} = (R \setminus \mathfrak{p})^{-1}\mathfrak{p}$; hence the process is called localization. The next result for modules over local rings is used several times below.

Fact 2.1.6 (Nakayama's lemma [11, Lemma 4.8])**.** Let $(R, \mathfrak{m})$ be a local ring and let M be a finitely generated R-module. Then the following conditions are equivalent:

(a) $M = 0$

(b) $\mathfrak{m}M = M$

(c) $M/\mathfrak{m}M = 0$.

Next, we introduce the notion of 'regular sequences.' These are used below to define the 'depth' of rings and modules.

Definition 2.1.7. An element $r \in R$ is a *non-zero divisor* on M if $m \in M$ and $rm = 0$ implies $m = 0$. Set

$$\operatorname{NZD}_R(M) := \{r \in R : r \text{ is a non-zero divisor on } M\}\,.$$

Equivalently, $r \in R$ is a non-zero divisor if the multiplication by r map $M \xrightarrow{\cdot r} M$ is injective.

Definition 2.1.8. An element $r \in R$ is M*-regular*, if

(a) $r \in \operatorname{NZD}_R(M)$ and

(b) $rM \neq M$.

A sequence $r_1, \dots, r_n \in R$ is *regular on* M, or M*-regular* for short, if

(a) r_1 is M-regular, and

(b) r_i is $\frac{M}{(r_1,\dots,r_{i-1})M}$-regular for all $i \in \{2, 3, \dots, n\}$.

Remark. Note that for $r_1, \dots, r_i \in R$ with $i \geq 2$, we have

$$\frac{M}{(r_1,\dots,r_i)M} \cong \frac{M/(r_1,\dots,r_{i-1})M}{(r_1,\dots,r_i)M/(r_1,\dots,r_{i-1})M} \cong \frac{M/(r_1,\dots,r_{i-1})M}{r_iM/(r_1,\dots,r_{i-1})M}$$

where the first isomorphism follows from the third isomorphism theorem for modules. Hence $r_iM/(r_1,\dots,r_{i-1})M \neq M/(r_1,\dots,r_{i-1})M$ if and only if $M/(r_1,\dots,r_i)M \neq 0$ if and only if $M \neq (r_1,\dots,r_i)M$. This gives the following equivalent definition of M-regular sequences.

Fact 2.1.9 ([14, Discussion II.C.5.1.])**.** The sequence $r_1, \dots, r_n \in R$ is *M-regular* if and only if

(a) $r_1 \in \mathrm{NZD}_R(M)$,

(b) $r_i \in \mathrm{NZD}_R(M/(r_1, \dots, r_{i-1})M)$ for all $i \in \{2, 3, \dots, n\}$, and

(c) $(r_1, \dots, r_n)M \neq M$.

Remark. Condition (c) in Fact 2.1.9 is automatically satisfied if $(R, \mathfrak{m})$ is local, $r_1, \dots, r_n \in \mathfrak{m}$, and $M \neq 0$ is finitely generated: by Nakayama's lemma, we have $(r_1, \dots, r_n)M \subseteq \mathfrak{m}M \neq M$ since $M \neq 0$.

Definition 2.1.10. An M-regular sequence $r_1, \dots, r_n \in I \leq R$ is *maximal in I* if for any $r_{n+1} \in I$, the sequence $r_1, \dots, r_n, r_{n+1}$ is not M-regular.

2.2 Exact Sequences and Projective Modules

In this section, we state some definitions and facts to describe the equivalent conditions in Fact 2.2.8 to define 'projective modules.' This is a preliminary section for Section 2.3.

Definition 2.2.1. Let M' and M'' be R-modules. Then a sequence of R-module homomorphisms

$$M \xrightarrow{f} M' \xrightarrow{g} M''$$

is *exact* (*at M'*) if $\operatorname{Im} f = \operatorname{Ker} g$. In general, a sequence of R-module homomorphisms

$$\dots \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \xrightarrow{d_{i-1}} \dots$$

is *exact* if $\operatorname{Im} d_{i+1} = \operatorname{Ker} d_i$ for all relevant i.

Fact 2.2.2. We have the following facts: Let $f : M \to M'$ and $g : M' \to M''$ be R-module homomorphisms.

(a) A sequence $0 \longrightarrow M \xrightarrow{\ f\ } M'$ is exact if and only if f is injective.

(b) A sequence $M' \xrightarrow{\ g\ } M'' \longrightarrow 0$ is exact if and only if g is surjective.

(c) A sequence $0 \longrightarrow M \xrightarrow{\ f\ } M' \xrightarrow{\ g\ } M'' \longrightarrow 0$ is exact if and only if f is injective, g is surjective, and $\operatorname{Im} f = \operatorname{Ker} g$.

Definition 2.2.3. When the sequence in Fact 2.2.2 (c) above is exact, it is called a *short exact sequence.*

Fact 2.2.4 (The short five lemma)**.** Given the following commutative diagram of R-module homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B & \xrightarrow[g']{} & C' & \longrightarrow & 0 \end{array}$$

we have:

(a) If α and γ are injective, so is β.

(b) If α and γ are surjective, so is β.

(c) If α and γ are isomorphisms, so is β.

Definition 2.2.5. Given R-modules A, B, and C, a short exact sequence

$$0 \longrightarrow A \xrightarrow{\ \psi\ } B \xrightarrow{\ \phi\ } C \longrightarrow 0$$

is said to be *split* if there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\varepsilon} & A \oplus C & \xrightarrow{\rho} & C & \longrightarrow & 0 \\
 & & \downarrow{\cong} & & \downarrow{\Gamma} & & \downarrow{\cong} & & \\
0 & \longrightarrow & A' & \xrightarrow[\psi]{} & B & \xrightarrow[\phi]{} & C' & \longrightarrow & 0
\end{array}$$

where ε and ρ are natural injection and surjection, respectively. In this case Γ is an isomorphism by the short five lemma, hence $B \cong A \oplus C$.

Notation 2.2.6. Let A, B, N be R-modules. We denote the set of all R-module homomorphisms from A to B by

$$\mathrm{Hom}_R(A, B) = \{f : A \to B \mid f \text{ is an } R\text{-module homomorphism}\}$$

which is an R-module since R is commutative. For each $f \in \mathrm{Hom}_R(A, B)$, we define

$$\begin{aligned} f_* := \mathrm{Hom}_R(N, f) : \mathrm{Hom}_R(N, A) &\to \mathrm{Hom}_R(N, B) \\ \phi &\mapsto f \circ \phi \end{aligned}$$

Then f_* is an R-module homomorphism.

Fact 2.2.7. If N is an R-module, then $\mathrm{Hom}_R(N, -)$ is a 'covariant functor,' i.e.,

(a) $\mathrm{Hom}_R(N, -)$ respects identity maps: $\mathrm{Hom}_R(N, id_M) = id_{\mathrm{Hom}_R(N,M)}$, and

(b) $\mathrm{Hom}_R(N, -)$ respects compositions: for all R-module homonomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we have

$$\mathrm{Hom}_R(N, g \circ f) = \mathrm{Hom}_R(N, g) \circ \mathrm{Hom}_R(N, f).$$

Equivalently, we have $(g \circ f)_* = g_* \circ f_*$, i.e., the following diagram commutes.

$$\begin{array}{llcl} \mathrm{Hom}_R(N,-): & \mathrm{Hom}_R(N,A) & \xrightarrow{f_*} & \mathrm{Hom}_R(N,B) \\ & & \searrow^{(g\circ f)_*} & \downarrow g_* \\ & & & \mathrm{Hom}_R(N,C) \end{array}$$

Fact 2.2.8 ([3, Proposition 10.30])**.** Let P be an R-module. Then the following are equivalent:

(a) The covariant functor $\mathrm{Hom}_R(P,-)$ transforms short exact sequences into short exact sequences.

(b) For any R-modules M and N, if $M \xrightarrow{\varphi} N \longrightarrow 0$ is exact, then every R-module homomorphism from P into N "lifts" to an R-module homomorphism into M, i.e., given $f \in \mathrm{Hom}_R(P,N)$, there is a lift $F \in \mathrm{Hom}_R(P,M)$ making the following diagram commute:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{\exists F} & \downarrow f & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

(c) If P is a quotient of the R-module M, then P is isomorphic to a direct summand of M, i.e., every short exact sequence $0 \to L \to M \to P \to 0$ splits.

(d) P is a direct summand of a free R-module.

Definition 2.2.9. An R-module P satisfying the equivalent conditions in Fact 2.2.8 is called *projective.*

2.3 Ext via Projective Resolutions

In this section, we define the 'Ext functor' using 'projective resolutions.'

Definition 2.3.1. A *chain complex* or an *R-complex* is a sequence of R-module homomorphisms

$$C = \cdots \xrightarrow{\partial_{i+1}^C} C_i \xrightarrow{\partial_i^C} C_{i-1} \xrightarrow{\partial_{i-1}^C} \cdots$$

such that $\partial_i^C \circ \partial_{i+1}^C = 0$ for all $i \in \mathbb{Z}$. The map ∂_i^C is called the ith *differential* of C. We say C_i is the module in *degree* i in the R-complex C. The i^{th} *homology module of* C is the R-module

$$H_i(C) := \frac{\operatorname{Ker}(\partial_i^C)}{\operatorname{Im}(\partial_{i+1}^C)}.$$

Definition 2.3.2. Suppose that we are given the following chain complex

$$P^+ = \cdots \xrightarrow{\partial_{i+1}^P} P_i \xrightarrow{\partial_i^P} P_{i-1} \xrightarrow{\partial_{i-1}^P} \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \longrightarrow 0$$

where P_i's are R-modules. If the sequence is exact and all P_i's are projective, then we say P^+ is a *projective resolution of* M *over* R, or simply an R*-projective resolution of* M. Given the R-projective resolution P^+ above, the *truncated projective resolution* of M associate to P^+ is the chain complex

$$P = \cdots \xrightarrow{\partial_{i+1}^P} P_i \xrightarrow{\partial_i^P} P_{i-1} \xrightarrow{\partial_{i-1}^P} \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0$$

which is exact at P_i for all $i > 0$, but not necessarily exact at P_0.

Notation 2.3.3. Let A, B be R-modules. Given $f \in \operatorname{Hom}_R(A, B)$, we denote

$$f^* := \operatorname{Hom}_R(f, N) : \operatorname{Hom}_R(B, N) \to \operatorname{Hom}_R(A, N)$$
$$\phi \mapsto \phi \circ f$$

which is an R-module homomorphism.

Fact 2.3.4. If N is an R-module, then $\mathrm{Hom}_R(-, N)$ is a 'contravariant functor,' i.e.,

(a) $\mathrm{Hom}_R(-, N)$ respects identity maps: $\mathrm{Hom}_R(id_M, N) = id_{\mathrm{Hom}_R(M,N)}$, and

(b) $\mathrm{Hom}_R(-, N)$ respects compositions but with the order reversed: for all R-module homonomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, we have

$$\mathrm{Hom}_R(g \circ f, N) = \mathrm{Hom}_R(f, N) \circ \mathrm{Hom}_R(g, N).$$

Equivalently, we have $(g \circ f)^* = f^* \circ g^*$, i.e., the following diagram commutes:

$$\begin{array}{lccc} \mathrm{Hom}_R(-, N): & \mathrm{Hom}_R(A, N) & \xleftarrow{f^*} & \mathrm{Hom}_R(B, N) \\ & & \nwarrow^{(g\circ f)^*} & \uparrow g^* \\ & & & \mathrm{Hom}_R(C, N) \end{array}$$

Fact 2.3.5 ([3, Theorem 10.33])**.** Let A, B, and C be R-modules. If

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact, then so is

$$0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g^*} \mathrm{Hom}_R(B, N) \xrightarrow{f^*} \mathrm{Hom}_R(A, N) \ .$$

Definition 2.3.6. Let P^+ be a R-projective resolution of M as in Definition 2.3.2, and let N be an R-module. We define the R-complex $\mathrm{Hom}_R(P^+, N)$ as follows: For all $i \geq 0$, we set $P_i^* := \mathrm{Hom}_R(P_i, N)$ and set $M^* := \mathrm{Hom}_R(M, N)$. Then we define

$\mathrm{Hom}_R(P^+, N) = (P^+)^*$ as

$$(P^+)^* := 0 \longrightarrow M^* \xrightarrow{\tau^*} P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} \cdots \xrightarrow{(\partial_{i-1}^P)^*} P_{i-1}^* \xrightarrow{(\partial_i^P)^*} P_i^* \xrightarrow{(\partial_{i+1}^P)^*} \cdots .$$

For the truncated projective resolution P of P^+, we define $\mathrm{Hom}_R(P, N) = P^*$ as

$$P^* := 0 \longrightarrow P_0^* \xrightarrow{(\partial_1^P)^*} P_1^* \xrightarrow{(\partial_2^P)^*} \cdots \xrightarrow{(\partial_{i-1}^P)^*} P_{i-1}^* \xrightarrow{(\partial_i^P)^*} P_i^* \xrightarrow{(\partial_{i+1}^P)^*} \cdots .$$

Remark. Let P^+ be an R-projective resolution of M. The sequences $(P^+)^*$ and P^* defined in Definition 2.3.6 are indeed R-complexes since $\partial_i^P \circ \partial_{i+1}^P = 0$ for $i \geq 0$ by definition of chain complex, and by Fact 2.2.7, we get

$$(\partial_{i+1}^P)^* \circ (\partial_i^P)^* = (\partial_i^P \circ \partial_{i+1}^P)^* = 0^* = 0, \text{ and } (\partial_1^P)^* \circ (\tau)^* = (\tau \circ \partial_1^P)^* = 0^* = 0.$$

Notation 2.3.7. Let P^+ be an R-projective resolution of M. Set $(P^*)_{-i} := P_i^*$ for all $i \geq 0$. and $\partial_{-i+1}^{P^*} := (\partial_i^P)^*$ for all $i \geq 1$. Graphically, we get

$$\begin{array}{ccccccccccccc}
P^* = 0 & \longrightarrow & P_0^* & \xrightarrow{(\partial_1^P)^*} & P_1^* & \xrightarrow{(\partial_2^P)^*} & \cdots & \xrightarrow{(\partial_{i-1}^P)^*} & P_{i-1}^* & \xrightarrow{(\partial_i^P)^*} & P_i^* & \xrightarrow{(\partial_{i+1}^P)^*} & \cdots \\
 & & \| & & \| & & & & \| & & \| & & \\
P^* = 0 & \longrightarrow & (P^*)_0 & \xrightarrow{\partial_0^{P^*}} & (P^*)_{-1} & \xrightarrow{\partial_{-1}^{P^*}} & \cdots & \xrightarrow{\partial_{-i+2}^{P^*}} & (P^*)_{-i+1} & \xrightarrow{\partial_{-i+1}^{P^*}} & (P^*)_{-i} & \xrightarrow{\partial_{-i}^{P^*}} & \cdots
\end{array}$$

All we are doing here is defining the degree of modules in P^* so that as we move along the differentials, the degree of the module goes down by 1. Also, we define $(\partial^{P^*})_i$ to be the 0 map for all $i \leq -1$.

Definition 2.3.8. Let P^+ be an R-projective resolution of M. For all $i \leq 0$, we define the *Ext module* by

$$\mathrm{Ext}_R^i(M, N) := H_{-i}(P^*) = \frac{\mathrm{Ker}(\partial_{-i}^{P^*})}{\mathrm{Im}(\partial_{-i+1}^{P^*})} = \frac{\mathrm{Ker}((\partial_{i+1}^P)^*)}{\mathrm{Im}((\partial_i^P)^*)} .$$

Fact 2.3.9. Let P^+ be a projective resolution of M. By Fact 2.3.5, the rows in the following commutative diagram are exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & M^* & \xrightarrow{\tau^*} & P_0^* & \xrightarrow{(\partial_1^P)^*} & P_1^* \\ & & \| & & \| & & \| \\ 0 & \longrightarrow & M^* & \xrightarrow{\tau^*} & (P^*)_0 & \xrightarrow{\partial_0^{P^*}} & (P^*)_{-1} \end{array}$$

Then we have

$$\begin{aligned} \operatorname{Ext}_R^0(M,N) = H_0(P^*) &= \frac{\operatorname{Ker}(\partial_0^{P^*})}{\operatorname{Im}(0 \to (P^*)_0)} && \text{(by Definition 2.3.8)} \\ &= \operatorname{Ker}(\partial_0^{P^*}) && \text{(since } \operatorname{Im}(0 \to (P^*)_0) = \{0\}) \\ &= \operatorname{Im}(\tau^*) && \text{(by the exactness)} \\ &\cong M^* && \text{(since } \tau^* \text{ is injective)} \\ &= \operatorname{Hom}_R(M,N). && \text{(by definition of } M^*) \end{aligned}$$

For all $i \leq -1$, the map $\partial_i^{P^*}$ is the 0 map whose image and kernel are both $\{0\}$ since its domain is the 0 module. Thus we get $\operatorname{Ext}_R^i(M,N) = 0$ for all $i \leq -1$.

Remark. $\operatorname{Ext}_R^i(M,N)$ exists and is well-defined, i.e., M has a projective resolution and $\operatorname{Ext}_R^i(M,N)$ is independent of the choice of the projective resolution of M; see [11, Theorem 5.2].

2.4 Tensor Products of Modules and Algebras

We introduce the notion of 'tensor product' and give a case where it behaves pretty well with the Ext modules.

Definition 2.4.1. Given R-modules A, B, and C, a function $f : A \times B \to C$ is called

R-*bilinear* if for all $a, a' \in A$, $b, b' \in B$, and $r \in R$, we have

$$f(a + a', b) = f(a, b) + f(a', b)$$
$$f(a, b + b') = f(a, b) + f(a, b')$$
$$f(ar, b) = f(a, rb) = rf(a, b).$$

Let F be the free abelian group on the set $A \times B$, and let K be the subgroup of F generated by all elements of the form

(1) $(a + a', b) - (a, b) - (a', b)$

(2) $(a, b + b') - (a, b) - (a, b')$

(3) $(ra, b) - (a, rb)$

The quotient F/K is denoted by $A \otimes_R B$. 'Simple tensors' in $A \otimes_R B$ are the elements $a \otimes b := (a, b) + K \in F/K$. Then $A \otimes_R B$ is an R-module and the natural injection $\iota : A \times B \to A \otimes_R B$ given by $(a, b) \mapsto a \otimes b$ is R-bilinear.

Let k be a field. We can also define the tensor product of 'k-algebras' which is natural for our setting when $R = k[V]$, a polynomial ring over k whose variables are the vertices of a given graph.

Definition 2.4.2. Let R be a commutative ring with identity which is also a k-vector space. Then R is a k-*algebra* if for all $x, y \in R$ and $a, b \in k$, we have $(ax) \cdot (by) = (ab) \cdot (xy)$.

Remark. Let R and S be k-algebras. Then R and S are k-modules, hence $R \otimes_k S$ is a k-module. In fact, multiplication on $R \otimes_k S$ given by $(r \otimes s) \cdot (r' \otimes s') = rr' \otimes ss'$ turns $R \otimes_k S$ into a k-algebra.

Example 2.4.3. Let $R = k[X]$ and $S = k[Y]$. Then both R and S are k-algebras. Also, one can show that $R \otimes_k S \cong k[X, Y]$ which is also a k-algebra. In general, if $R = k[X_1, \ldots, X_n]$ and $S = k[Y_1, \ldots, Y_m]$, then we have

$$k[X_1, \ldots, X_n, Y_1, \ldots, Y_m] \cong R \otimes_k S.$$

Example 2.4.4. Let k be a field, $R = k[X_1, \ldots, X_n]$ and $S = k[Y_1, \ldots, Y_m]$ for some $m, n \in \mathbb{N}$. Let $I \leq R$ and $J \leq S$ be monomial ideals. Then we can treat $I + J$ as a monomial ideal in $k[X_1, \ldots, X_n, Y_1, \ldots, Y_m] \cong R \otimes_k S$. Furthermore, we can show that

$$\frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]}{I + J} \cong \frac{R}{I} \otimes_k \frac{S}{J}$$

as k-algebras: note that every element in $k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]/(I + J)$ can be written as the sum $\sum_{i=1}^{t} r_i \overline{f_i g_i}$ where $r_i \in k$, $f_i \in R$ and $g_i \in S$ for all $i \in [t]$. One can show that

$$L : \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_m]}{I + J} \to \frac{R}{I} \otimes_k \frac{S}{J}$$

given by

$$L\left(\sum_{i=1}^{t} r_i \overline{f_i g_i}\right) = \sum_{i=1}^{t} r_i \left(\overline{f_i} \otimes_k \overline{g_i}\right)$$

is an isomorphism.

Now we state a fact that is used in Theorem 3.3.9:

Fact 2.4.5 ([13, Proposition A.1.5])**.** Let R and S be k-algebras. Let B and B' be R-modules where B is finitely generated, and let C and C' be S-modules such that C is finitely generated. Then for all $i \geq 0$, there are $R \otimes_k S$-module isomorphisms

$$\operatorname{Ext}^i_{R \otimes_k S}(B \otimes_k C, B' \otimes_k C') \cong \bigoplus_{j=0}^{i} \left(\operatorname{Ext}^j_R(B, B') \otimes_k \operatorname{Ext}^{i-j}_S(C, C')\right).$$

Example 2.4.6. With the same assumptions in Fact 2.4.5, we have

$$\begin{aligned}\operatorname{Ext}^0_{R\otimes_k S}(B\otimes_k C, B'\otimes_k C') &\cong \bigoplus_{j=0}^{0}\left(\operatorname{Ext}^j_R(B,B')\otimes_k \operatorname{Ext}^{i-j}_S(C,C')\right)\\ &= \operatorname{Ext}^0_R(B,B')\otimes_k \operatorname{Ext}^0_S(C,C').\end{aligned}$$

2.5 Krull Dimension, Depth, and Type

In this section, we define three invariants: 'Krull dimension,' 'depth,' and 'type.' The Krull dimension and depth are used to define the 'Cohen-Macaulay' property, and the type of a module is the main invariant that we investigate in this thesis.

Definition 2.5.1. We define the following sets:

(a) The *prime spectrum of* R is the set $\operatorname{Spec}(R) := \{\text{prime ideals in } R\}$.

(b) The *variety of* I is the set $V(I) := \{\mathfrak{p}\in\operatorname{Spec}(R) : I\subseteq\mathfrak{p}\}$.

(c) The *support of* M is the set $\operatorname{supp}_R(M) := \{\mathfrak{p}\in\operatorname{Spec}(R) : M_\mathfrak{p}\neq 0\}$.

Fact 2.5.2. It is not hard to show that $\operatorname{supp}_R(R) = \operatorname{Spec}(R)$ and $\operatorname{supp}_R(R/I) = V(I)$.

Definition 2.5.3. The *Krull dimension of* M is

$$\dim_R(M) := \sup\left\{n\in\mathbb{N}_0 : \exists \text{ a chain } \mathfrak{p}_0\subsetneq\mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_n \text{ in } \operatorname{supp}_R(M)\right\}.$$

Using Fact 2.5.2, we can compute the Krull dimension of R and R/I as follows.

Fact 2.5.4. We compute the Krull dimension of R and R/I by

(a) $\dim(R) := \dim_R(R) = \sup\{n : \exists \text{ a chain } \mathfrak{p}_0\subsetneq\mathfrak{p}_1\subsetneq\cdots\subsetneq\mathfrak{p}_n \text{ in } \operatorname{Spec}(R)\}$.

(b) $\dim(R/I) := \dim_R(R/I) = \sup\{n : \exists \text{ a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } V(I)\}$.

Example 2.5.5. Let $R_1 := k[X_1, \ldots, X_d]_{\langle X_1,\ldots,X_d\rangle}$ and $R_2 = k[X,Y]_{\langle X,Y\rangle}/\langle X^2, XY\rangle$. First, note that the Krull dimension of $k[X_1, \ldots, X_d]$ is d, given by the chain

$$0 \subsetneq \langle X_1\rangle \subsetneq \langle X_1, X_2\rangle \subsetneq \cdots \subsetneq \langle X_1, \ldots, X_d\rangle.$$

By Fact 2.1.5, we must have $\dim(R_1) \leq \dim(k[X_1, \ldots, X_d]) = d$. But also, the prime ideals in the chain above survive the localization since $\langle X_1, \ldots, X_k\rangle \subseteq \langle X_1, \ldots, X_d\rangle$ for $k \in [d]$. Thus we have $\dim(R_1) = d$ using the same chain above, but localized. For R_2, consider $k[X,Y]/\langle X^2, XY\rangle$. Since we have $\langle X^2, XY\rangle = \langle X\rangle \cap \langle X^2, Y\rangle$, by Fact 2.6.15 we get $\dim(k[X,Y]/\langle X^2, XY\rangle) = 2 - 1 = 1$. Again by Fact 2.1.5, we have $\dim(R_2) \leq \dim(k[X,Y]/\langle X^2, XY\rangle) = 1$. On the other hand, the chain of prime ideals $\langle \overline{X}\rangle \subsetneq \langle \overline{X}, \overline{Y}\rangle$ shows that $\dim(R_2) \geq 1$.

Next, we define the 'depth' of a finitely generated R-module M using the length of maximal M-regular sequences.

Assumptions. For the remainder of the section, we assume that R is Noetherian and M is finitely generated.

Fact 2.5.6 ([2, Theorem 1.2.5])**.** Let $I \leq R$ be an ideal such that $IM \neq M$. Then all maximal M-regular sequences in I have the same length n given by

$$n = \min\left\{i : \operatorname{Ext}^i_R(R/I, M) \neq 0\right\}.$$

By Fact 2.5.6, the following definition of depth of M in I is now well-defined.

Definition 2.5.7. Given an ideal $I \leq R$ such that $IM \neq M$, the *depth* of M in I

(also called the *grade of I on M*) is

$$\begin{aligned} \operatorname{depth}_R(I, M) &:= \text{“length of a maximal } M\text{-regular sequences in } I\text{”} \\ &= \min\left\{n : \operatorname{Ext}^n_R(R/I, M) \neq 0\right\}. \qquad \text{(Fact 2.5.6)} \end{aligned}$$

If $IM = M$, then we set $\operatorname{depth}_R(I, M) := \infty$.

Notation 2.5.8. If $(R, \mathfrak{m})$ is local, then we set $\operatorname{depth}_R(M) := \operatorname{depth}_R(\mathfrak{m}, M)$. If $M \neq 0$, then $\mathfrak{m}M \neq M$ by Nakayama's lemma (Fact 2.1.6). If $M = R$ or $M = R/I$, then we omit the subscript R and write $\operatorname{depth}(R) := \operatorname{depth}_R(R)$ and $\operatorname{depth}(R/I) := \operatorname{depth}_R(R/I)$.

Example 2.5.9. Let R_1 and R_2 be the rings from Example 2.5.5. The sequence $X_1, \ldots, X_d$ in R_1 forms a regular sequence. It is maximal since $R_1/\langle X_1, \ldots, X_d\rangle \cong k$ which has no regular element since every element is either 0 or a unit. Thus we get $\operatorname{depth}(R_1) = d$. Alternatively, one can check directly that $\operatorname{Ext}^i_{R_1}(k, R_1) = 0$ for all $i < d$ and $\operatorname{Ext}^d_{R_1}(k, R_1) \cong k$. For the depth of R_2, we show that any element $\overline{r} \in R_2$ is either 0, or a unit, or a zero-divisor; hence $\operatorname{depth}(R_2) = 0$. If $\overline{r} \notin \langle \overline{X}, \overline{Y}\rangle$, then $\overline{r}$ is a unit in R_2 by localization. So suppose that $0 \neq \overline{r} \in \langle \overline{X}, \overline{Y}\rangle$. Then there are elements $\overline{f}, \overline{g} \in R_2$ such that $\overline{r} = \overline{fX} + \overline{gY}$. Since we have $\overline{X}\overline{r} = \overline{X}(\overline{fX} + \overline{gY}) = \overline{fX}^2 + \overline{g}\overline{XY} = \overline{0}$ in R_2, the element $\overline{r}$ must be a zero-divisor. Thus R_2 has no regular elements, and therefore we conclude that $\operatorname{depth}(R_2) = 0$.

Now, we are ready to define ‘Cohen-Macaulay-ness’ if R is a local ring. The importance of ‘Cohen-Macaulay modules’ is described, e.g., in [10, p. 175], [2, p. 57].

Definition 2.5.10. Let $(R, \mathfrak{m})$ be a local ring, and assume $M \neq 0$. We say M is *Cohen-Macaulay* if $\dim_R(M) = \operatorname{depth}_R(M)$. If R is a Cohen-Macaulay module over itself, then we say R is a *Cohen-Macaulay ring*.

Example 2.5.11. Let R_1 and R_2 be the rings from Example 2.5.9, and set $R = k[X, Y]$. From Examples 2.5.5 and 2.5.9, we get

$$\dim(R_1) = d = \operatorname{depth}(R_1) \quad \text{and} \quad \dim(R_2) = 1 \neq 0 = \operatorname{depth}(R_2).$$

Hence R_1 is Cohen-Macaulay while R_2 is not Cohen-Macaulay.

Finally, we present the definition of 'Cohen-Macaulay type' (or just 'type') for a special case when R is local. 'Type' will be redefined for the case when R is a polynomial ring over a field in Section 2.7.

Definition 2.5.12. Let $(R, \mathfrak{m}, k)$ be a local ring. Assume $\operatorname{depth}_R(M) = n$. The *type* of M is the positive integer

$$\operatorname{type}_R(M) := \dim_k(\operatorname{Ext}^n_R(R/\mathfrak{m}, M)) = \dim_k(\operatorname{Ext}^n_R(k, M)).$$

Example 2.5.13. Let $R = k[X_1, \ldots, X_d]_{\langle X_1,\ldots,X_d\rangle}$. Then we have $\operatorname{type}_R(R) = 1$, which can be shown using Facts 2.7.4, 2.7.7, and 2.7.10 below, or by checking directly that $\operatorname{Ext}^d_R(k, R) \cong k$.

2.6 Parametric Decomposition

In this section, we discuss a way to compute the type of a quotient ring if the relative ideal has a 'parametric decomposition.' For more details on monomial ideals, see [10].

Assumptions. Unless stated otherwise, in this section $R := k[X_1, \ldots, X_d]$ where k is a field with $d \in \mathbb{N}$, and $\mathfrak{X} := (X_1, \ldots, X_d)R$.

Definition 2.6.1. A *monomial* in the elements $X_1, \dots, X_d \in R$ is an element in R of the form $X_1^{n_1} \cdots X_d^{n_d}$ where $n_i \in \mathbb{N}_0$ for all $i \in [d]$. An ideal I in R is called *monomial ideal* if I can be generated by a (possibly empty) set of monomials in R.

Example 2.6.2. The trivial ideals 0 and R are monomial ideals since $(\emptyset)R = 0$ and $R = (1)R = (X_1^0 \cdots X_d^0)R$. If $d = 3$, the elements X_1X_2, $X_1^3X_2X_3^6$, and $X_1^2X_3^4$ are monomials in R.

The following fact is a useful tool when dealing with monomial ideals, used in Proposition 2.6.28.

Fact 2.6.3 ([10, Lemma 1.1.10])**.** Let $I \leq R$ be an ideal. Then I is a monomial ideal if and only if for all $f \in I$, every monomial occurring in f is in I.

Here is a handy, non-standard notation.

Notation 2.6.4. For $\underline{n} := (n_1, \dots, n_d) \in \mathbb{N}_0^d$, we set $\underline{X}^{\underline{n}} := X_1^{n_1} \cdots X_d^{n_d}$. For any $S \subseteq R$, we set $[\![S]\!] := \left\{\underline{X}^{\underline{n}} : \underline{X}^{\underline{n}} \in S, \underline{n} \in \mathbb{N}_0^d\right\}$ the set of all monomials in S.

The following fact is used in Lemma 2.6.27.

Fact 2.6.5 ([10, Theorem 1.1.4])**.** Let $I, J \leq R$ be monomial ideals.

(a) $I \subseteq J$ if an only if $[\![I]\!] \subseteq [\![J]\!]$.

(b) $I = J$ if and only if $[\![I]\!] = [\![J]\!]$.

The existence of 'm-irreducible decomposition' of monomial ideals, which will be used to define the 'parametric decomposition,' arise from the fact that every monomial ideal in a polynomial ring is generated by finitely many monomials [10, Theorem 1.3.1].

Definition 2.6.6. Let $I \leq R$. The *radical* of I is the ideal

$$\mathrm{rad}(I) := \{x \in R : \exists n \in \mathbb{N}\ s.t.\ x^n \in I\}.$$

The *monomial radical* of a monomial ideal $I \leq R$ is the monomial ideal m-rad$(I) :=$ $(\mathrm{rad}(I) \cap [\![R]\!])R$.

Definition 2.6.7. Let $I \leq R$ be a monomial ideal. We say I is *m-reducible* if there are monomial ideals $J, L \leq R$ such that $I = J \cap L$ with $J \neq I$ and $L \neq I$. The ideal I is *m-irreducible* if I is not m-reducible.

Remark. There is a notion of 'reducible' and 'irreducible' of ideals (without the 'm-') by assuming that the factors J, L above are not necessarily monomial ideals. But if R is a polynomial ring over a field, the two notions coincide for monomial ideals; see [10, Theorem 3.2.4].

The following fact describes precisely what are the m-irreducible ideals in R.

Fact 2.6.8 ([10, Theorem 3.1.4])**.** The monomial ideal 0 is m-irreducible. A non-zero monomial ideal $I \leq R$ is m-irreducible if and only if $I = (X_{i_1}^{a_1}, \ldots, X_{i_t}^{a_t})R$ for some $t \in [d]$ and $a_i \geq 1$ for all $i \in [t]$.

Definition 2.6.9. An *m-irreducible decomposition* of a monomial ideal $I \leq R$ is an expression $I = \bigcap_{i=1}^{n} J_i$ with $n \in \mathbb{N}$ such that $J_1, \ldots, J_n$ are m-irreducible monomial ideals of R.

Fact 2.6.10 ([10, Theorem 3.3.3])**.** Every monomial ideal $I \leq R$ has an m-irreducible decomposition.

Definition 2.6.11. Let $J \subsetneq R$ be a monomial ideal generated by $f_1, \ldots, f_k \in [\![R]\!]$ with m-irreducible decomposition $J = \bigcap_{i=1}^{n} I_i$. We say $f_1, \ldots, f_k$ is an *irredundant* *monomial generating sequence* for J if $\langle f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_k \rangle \neq J$ for all $i \in [k]$. We say $\bigcap_{i=1}^n I_i$ is *irredundant* if $\bigcap_{i \in [n] \setminus \{j\}} I_i \neq J$ for all $j \in [n]$. The generating sequence is *redundant* if it is not irredundant, and similarly for the decomposition.

The irredundant m-irreducible decomposition of monomial ideal is unique up to permutation, stated next.

Fact 2.6.12 ([10, Theorem 3.3.9])**.** Let $I \leq R$ be a monomial ideal with irredundant m-irreducible decompositions $\bigcap_{i=1}^m I_i$ and $\bigcap_{j=1}^n J_j$. Then $m = n$ and there is a permutation σ on $[n]$ so that $I_i = J_{\sigma(i)}$ for all $i \in [n]$.

One can compute an m-irreducible decomposition of a monomial ideal by applying the following fact repeatedly.

Fact 2.6.13 ([10, Lemma 3.1.3])**.** Let $I \leq R$ be a monomial ideal with monomial generating sequence $f_1, \ldots, f_k \in [\![R]\!]$. Suppose that there exists $l \in [d]$ such that $f_k = X_l^e g$ where $e \geq 1$, $g \in [\![R]\!]$, and $X_l^{e+1} \nmid f_k$. Set $I' := \langle f_1, \ldots, f_{k-1}, X_1^e \rangle$ and $I'' := \langle f_1, \ldots, f_{k-1}, g \rangle$. Then we have $I = I' \cap I''$ with $I' \neq I$ and $I'' \neq I$. In particular, I is m-reducible.

Example 2.6.14. Let $R = k[X_1, \ldots, X_7]$ and let $I = \langle X_1X_3, X_3X_5, X_5X_7 \rangle$. We will use Fact 2.6.13 to compute the irredundant m-irreducible decomposition of I:

$$\begin{aligned}
I &= \langle X_1X_3, X_3X_5, X_5X_7 \rangle \\
&= \langle X_1, X_3X_5, X_5X_7 \rangle \cap \langle X_3, X_3X_5, X_5X_7 \rangle && \text{(Fact 2.6.13)} \\
&= \langle X_1, X_3X_5, X_5X_7 \rangle \cap \langle X_3, \cancel{X_3X_5}, X_5X_7 \rangle && \text{(redundant generator)} \\
&= (\langle X_1, X_3, X_5X_7 \rangle \cap \langle X_1, X_5, X_5X_7 \rangle) \cap (\langle X_3, X_5 \rangle \cap \langle X_3, X_7 \rangle) && \text{(Fact 2.6.13)} \\
&= \langle X_1, X_3, X_5X_7 \rangle \cap \langle X_1, X_5, \cancel{X_5X_7} \rangle \cap \langle X_3, X_5 \rangle \cap \langle X_3, X_7 \rangle && \text{(redundant gen.)} \\
&= (\langle X_1, X_3, X_5 \rangle \cap \langle X_1, X_3, X_7 \rangle) \cap \langle X_1, X_5 \rangle \cap \langle X_3, X_5 \rangle \cap \langle X_3, X_7 \rangle && \text{(Fact 2.6.13)}
\end{aligned}$$

$$= \langle X_1, X_5\rangle \cap \langle X_3, X_5\rangle \cap \langle X_3, X_7\rangle$$

where the last equality holds since $\langle X_1, X_3, X_5\rangle \supset \langle X_1, X_5\rangle$ and $\langle X_1, X_3, X_7\rangle \supset \langle X_3, X_7\rangle$.

The next fact states that, in the monomial setting, Krull dimension can be computed using m-irreducible decompositions as in Fact 2.5.4.

Fact 2.6.15 ([10, Theorem 5.1.2])**.** Let $I \leq R$ be a monomial ideal with m-irreducible decomposition $I = \bigcap_{i=1}^{n} J_i$. Then $\dim(R/I) = d - m$ where m is the smallest number of generators needed for one of the J_i's.

Example 2.6.16. Recall the Krull dimension computation of R_2 in Example 2.5.5. The Krull dimension of $k[X, Y]/\langle X^2, XY\rangle$ is computed using Fact 2.6.15. The m-irreducible decomposition of the ideal $\langle X^2, XY\rangle$ is given by

$$\left\langle X^2, XY\right\rangle = \left\langle X^2, X\right\rangle \cap \left\langle X^2, Y\right\rangle = \langle X\rangle \cap \left\langle X^2, Y\right\rangle.$$

Since $k[X, Y]$ has 2 variables and the smallest number of generators for the ideals $\langle X\rangle$ and $\langle X^2, Y\rangle$ in the decomposition is 1, we get $\dim(k[X, Y]/\langle X^2, XY\rangle) = 2 - 1 = 1$.

Next, consider I and R from Example 2.6.14. The m-irreducible decomposition of I is given by

$$I = \langle X_1, X_5\rangle \cap \langle X_3, X_5\rangle \cap \langle X_3, X_7\rangle.$$

Hence the smallest number of generators of the ideals in the decomposition is 2. Since R has 7 variables, we get $\dim(R/I) = 7 - 2 = 5$.

Next, we define 'parametric decomposition' of monomial ideals after defining 'parameter ideals.'

Definition 2.6.17. A *parameter ideal* in R is an ideal of the form $\langle X_1^{a_1}, \ldots, X_d^{a_d} \rangle$ with $a_i \geq 1$ for all $i \in [d]$. For $\underline{X}^{\underline{n}} \in R$ with $\underline{n} \in \mathbb{N}_0^d$, we denote

$$P_R(\underline{X}^{\underline{n}}) := \left\langle X_1^{n_1+1}, \ldots, X_d^{n_d+1} \right\rangle$$

which is a parameter ideal by construction. If a monomial ideal $I \leq R$, has an (irredundant) m-irreducible decomposition $I = \bigcap_{i=1}^n J_i$ with each J_i being a parameter ideal in R, then we call $\bigcap_{i=1}^n J_i$ an (irredundant) *parametric decomposition* of I in R.

Example 2.6.18. Let $R = k[X_1, X_2, X_3, X_4]$. If $\underline{n} = (2, 3, 1, 2)$, then we have $P_R(\underline{X}^{\underline{n}}) = \langle X_1^3, X_2^4, X_3^2, X_4^3 \rangle$. The ideal $\langle X_1^5, X_2^2, X_3^2, X_4^3 \rangle$ is a parameter ideal, but $\langle X_1, X_3, X_4^3 \rangle$ is not a parameter ideal since it is missing the variable X_2.

Example 2.6.19. The ideal I in Example 2.6.14. does not have a parametric decomposition in $k[X_1, \ldots, X_7]$. Indeed, the irredundant m-irreducible decomposition

$$I = \langle X_1, X_5 \rangle \cap \langle X_3, X_5 \rangle \cap \langle X_3, X_7 \rangle$$

is not a parametric decomposition. Since every parametric decomposition is an m-irreducible decomposition, the uniqueness in Fact 2.6.12 implies that I does not have a parametric decomposition.

On the other hand, let $R = k[X, Y, Z]$ and set $J = \langle XY, YZ, X^2, Y^2, Z^2 \rangle \leq R$. The irredundant m-irreducible decomposition of J is $J = \langle X, Y^2, Z \rangle \cap \langle X^2, Y, Z^2 \rangle$ which is a parametric decomposition in R.

When $I \leq R$ is a monomial ideal with a parametric decomposition, we can compute the type of a quotient ring R/I by looking at the parametric decomposition of I. For this purpose, we define the 'I-corner' elements in R.

Definition 2.6.20. Let $I \leq R$ be a monomial ideal. A monomial $f \in [\![R]\!]$ is an *I-corner element* if $f \notin I$ and $X_i f \in I$ for all $i \in [d]$. We denote the set of all I-corner elements in R by $C_R(I)$.

Example 2.6.21. Let $R = k[X, Y]$ and $I = \langle Y^3, X^2Y^2, X^3Y, X^5 \rangle$. Consider the monomials XY^2, X^2Y, and X^4. These monomials are not in I as no monomial generators divide them in R. Since we have

$$\begin{aligned} X \cdot XY^2 &= X^2Y^2 \in I & \qquad Y \cdot XY^2 &= XY^3 \in I \\ X \cdot X^2Y &= X^3Y \in I & \qquad Y \cdot X^2Y &= X^2Y^2 \in I \\ X \cdot X^4 &= X^5 \in I & \qquad Y \cdot X^4 &= X \cdot X^3Y \in I \end{aligned}$$

the monomials XY^2, X^2Y, and X^4 are I-corner elements. In fact, these are the only I-corner elements. Indeed let

$$f \in [\![R]\!] \setminus [\![I]\!] = \left\{1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, X^4\right\}$$

be such that Xf and Yf are in I. Inspecting the elements of $[\![R]\!] \setminus [\![I]\!]$ we deduce that $f \in \{XY^2, X^2Y, X^4\}$. Hence we have $C_R(I) = \{XY^2, X^2Y, X^4\}$. Given a monomial X^aY^b for some $a, b \in \mathbb{N}$, we can graphically plot X^aY^b to the point (a, b) on the XY-plane. Figure 2.1 is a plot of I and $C_R(I)$ on the XY-plane, where the I-corner elements are the "circled **C**'s" in the figure.

It is straightforward to show that the I-corner elements are the maximal elements in $[\![R]\!] \setminus [\![I]\!]$ under the product order (componentwise order). The following theorem relates I-corner elements to the parametric decomposition of I if I has a parametric decomposition.

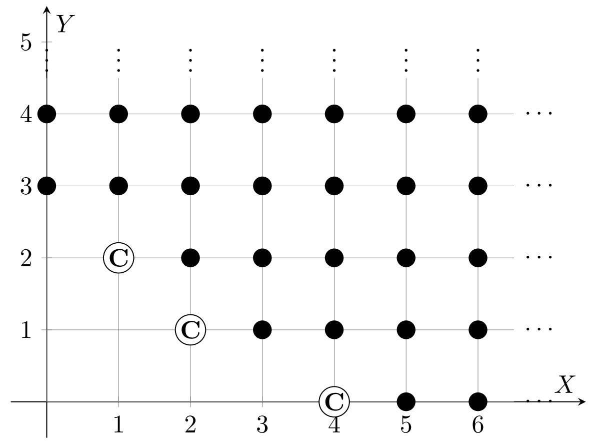


Figure 2.1: I and $C_R(I)$ plotted

Fact 2.6.22 ([10, Theorem 6.2.9 and Proposition 6.2.11])**.** Let $I \leq R$ be a monomial ideal and $f_1, \ldots, f_t \in [\![R]\!]$. Then $I = \bigcap_{i=1}^{t} P_R(f_i)$ is an irredundant parametric decomposition of I if and only if $C_R(I) = \{f_1, \ldots, f_t\}$ and m-rad$(I) = \mathfrak{X}$.

Example 2.6.23. Consider $R = k[X, Y]$ and $I = \langle Y^3, X^2Y^2, X^3Y, X^5 \rangle$ as in Example 2.6.21. Since m-rad$(I) = \langle X, Y \rangle = \mathfrak{X}$ and $C_R(I) = \{XY^2, X^2Y, X^4\}$, we get the following parametric decomposition of I immediately using Fact 2.6.22:

$$I = P_R(XY^2) \cap P_R(X^2Y) \cap P_R(X^4) = \left\langle X^2, Y^3 \right\rangle \cap \left\langle X^3, Y^2 \right\rangle \cap \left\langle X^5, Y \right\rangle.$$

Proposition 2.6.28 below is used in the proof of Fact 2.7.10. We start with the definition of the 'colon ideal.'

Definition 2.6.24. Let $I \leq R$ and $S \subseteq R$. The *ideal quotient* or *colon ideal* is the set

$$(I :_R S) := \{r \in R : rs \in I \ \forall s \in S\}.$$

The following fact shows that colon ideals are monomial ideals if the inputs are monomial ideals as well.

Fact 2.6.25 ([10, Theorem 2.5.1])**.** If $I, J \leq R$ are monomial ideals of R, then $(I :_R J)$ is a monomial ideal of R.

Given a monomial ideal $I \leq R$, the corner elements of I and the colon ideal $(I :_R \mathfrak{X})$ have a close relation shown next.

Fact 2.6.26 ([10, Proposition 6.2.3])**.** Let $I \leq R$ be a monomial ideal.

(a) $C_R(I) = [\![(I :_R \mathfrak{X})]\!] \setminus [\![I]\!]$.

(b) $C_R(I)$ is finite.

Lemma 2.6.27 ([10, Exercise 6.2.31])**.** *Let $I \leq R$ be a monomial ideal. Then*

$$(I :_R \mathfrak{X}) = I + (C_R(I))R.$$

Proof. ($\supseteq$) By definition of the I-corner elements, we have $\mathfrak{X}C_R(I) \subseteq I$. Hence by the definition of colon ideal, we get $(I :_R \mathfrak{X}) \supseteq C_R(I)$, which implies $(I :_R \mathfrak{X}) \supseteq (C_R(I))R$. Since $(I :_R \mathfrak{X}) \supseteq I$, we also have $(I :_R \mathfrak{X}) \supseteq I + (C_R(I))R$.

($\subseteq$) By Fact 2.6.5, it suffices to show that $[\![(I :_R \mathfrak{X})]\!] \subseteq [\![I + (C_R(I))R]\!]$. So, let $f \in [\![(I :_R \mathfrak{X})]\!]$. Then we have $f\mathfrak{X} \subseteq I$. If $f \in [\![I]\!]$, we are done. So suppose that $f \notin [\![I]\!]$. Since $f\mathfrak{X} \subseteq I$, we get $f \in C_R(I) = [\![C_R(I)]\!] \subseteq [\![I + (C_R(I))R]\!]$. $\square$

In the next result, we use Fact 2.6.26 (b) to see that the list of corner elements for a given monomial ideal is finite.

Proposition 2.6.28 ([17, Proposition 2.80])**.** *Let $I \leq R$ be a monomial ideal. If $f_1, \ldots, f_t$ are the distinct I-corner elements in R, then $\overline{f_1}, \ldots, \overline{f_t}$ is a basis of the vector space $(I :_R \mathfrak{X})/I$ over $R/\mathfrak{X} \cong k$.*

Proof. Since $(I :_R \mathfrak{X}) \leq R$ (by Fact 2.6.25), $(I :_R \mathfrak{X})$ is an R-module. In particular, we can treat $(I :_R \mathfrak{X})/I \leq R/I$ as a vector space over $k \subseteq R$. Furthermore, the condition $\mathfrak{X} \cdot (I :_R \mathfrak{X}) \subseteq I$ implies $\mathfrak{X} \cdot (I :_R \mathfrak{X})/I = 0$, so the action $\overline{f} \cdot \overline{r} = \overline{fr}$ gives a well-defined $(R/\mathfrak{X})$-module structure on $(I :_R \mathfrak{X})/I$ that is compatible with the isomorphism $R/\mathfrak{X} \cong k$. (See also the proof of Fact 2.7.10 below.)

Now let $\overline{f} \in (I :_R \mathfrak{X})/I$ with $f \in (I :_R \mathfrak{X})$. Since $C_R(I) = \{f_1, \ldots, f_t\}$ by the assumption, we get that $(I :_R \mathfrak{X}) = I + (f_1, \ldots, f_t)R$ by Lemma 2.6.27. Since we have

$$\frac{(I :_R \mathfrak{X})}{I} = \frac{I + (f_1, \ldots, f_t)R}{I}$$

we see that $(I :_R \mathfrak{X})$ is generated as an R-module by $\overline{f}_1, \ldots, \overline{f}_t$. So, there exist $r_1, \ldots, r_t \in R$ such that $\overline{f} = \overline{r_1}\overline{f_1} + \cdots + \overline{r_t}\overline{f_t}$ in R/I. By the definition of I-corner element, we have $X_i f_j \in I$ for any $i, j \in [t]$. So without loss of generality, we may assume that $r_i \in k$ for all $i \in [t]$. Thus $\overline{f_1}, \ldots, \overline{f_t}$ spans $(I :_R \mathfrak{X})/I$ over k.

Let $a_1, \ldots, a_t \in k$ be such that $a_1\overline{f_1} + \cdots + a_t\overline{f_t} = \overline{0}$ in $(I :_R \mathfrak{X})/I$. Assume by way of contradiction that not all the a_i's are 0. Since we have

$$\overline{0} = a_1\overline{f_1} + \cdots + a_t\overline{f_t} = \overline{a_1 f_1} + \cdots + \overline{a_t f_t} = \overline{a_1 f_1 + \cdots a_t f_t}\ ,$$

this implies that $a_1 f_1 + \cdots a_t f_t \in I$. Then the I-corner elements f_i such that $a_i \neq 0$ are monomials occurring in $a_1 f_1 + \cdots + a_t f_t$. Hence by Fact 2.6.3, we have $f_i \in I$ for all such i since I is a monomial ideal, contradicting the definition of I-corner element. □

We end this section with two more lemmas. Lemma 2.6.29 is used below in Theorem 3.2.9, and Lemma 2.6.30 is used in Theorems 3.3.7 and 3.3.9 for the Krull dimension computation. Both lemmas are corollary of Lemma 7.3.2 in [10].

Lemma 2.6.29 ([10, Lemma 7.3.2]). *Let* $U := \left\langle X_1^{k_1}, \ldots, X_d^{k_d} \right\rangle$ *where* $k_i \geq 2$ *for all*

$i \in [d]$. Let $D_1, \ldots, D_t \subseteq \{X_1, \ldots, X_d\}$ for some $t \geq 1$. Then we have

$$\left(\bigcap_{i=1}^{t} P_{D_i}\right) + U = \bigcap_{i=1}^{t} (P_{D_i} + U).$$

Lemma 2.6.30 ([10, Lemma 7.3.2])**.** *Let $R_1 = k[X_1, \ldots, X_{d_1}]$, $R_2 = k[Y_1, \ldots, Y_{d_2}]$. Let $I \leq R_1$ and $J \leq R_2$ be monomial ideals. Then we have*

$$\dim\left(\frac{R_1}{I} \otimes_k \frac{R_2}{J}\right) = \dim\left(\frac{R_1}{I}\right) + \dim\left(\frac{R_2}{J}\right).$$

Proof. Suppose that I and J have the following irredundant m-irreducible decompositions: $I = \bigcap_{i=1}^{n} I_i$ and $J = \bigcap_{j=1}^{m} J_i$. Set

$$R := R_1 \otimes_k R_2 = k[X_1, \ldots, X_{d_1}, Y_1, \ldots, Y_{d_2}].$$

Then by Lemma 7.3.2 in [10], the ideal $IR + JR \leq R$ has the m-irreducible decomposition

$$IR + JR = \bigcap_{i=1}^{n} \bigcap_{j=1}^{m} (I_i R + J_j R).$$

Then we have

$$\dim\left(\frac{R_1}{I} \otimes_k \frac{R_2}{J}\right) = \dim\left(\frac{R}{IR + JR}\right) \qquad \text{(Example 2.4.4)}$$

$$= (d_1 + d_2) - t \qquad \text{(Fact 2.6.15)}$$

where t is the smallest number of generators needed for one of the $I_i R + J_j R$'s. Since the irredundant generators of I and J share no variable in common, the same is true for I_i and J_j. Hence we get $t = t_1 + t_2$ where t_1 and t_2 are the smallest number of generators needed for I_i and J_j, respectively. Therefore, we have

$$\begin{aligned}
\dim\left(\frac{R_1}{I} \otimes_k \frac{R_2}{J}\right) &= (d_1 + d_2) - t \\
&= (d_1 + d_2) - (t_1 + t_2) \\
&= (d_1 - t_1) + (d_2 - t_2) \\
&= \dim\left(\frac{R_1}{I}\right) + \dim\left(\frac{R_2}{J}\right) && \text{(Fact 2.6.15)}
\end{aligned}$$

as desired. □

2.7 Cohen-Macaulayness and Type for Monomial Quotient Rings

In this section, we first define the concepts of depth and type for rings R/I where R is a polynomial ring over a field and $I \leq R$ is a monomial ideal. Also, regular sequences will be used to simplify the computation of the type.

Assumptions. Throughout this section, unless stated otherwise, assume that $R := k[X_1, \ldots, X_d]$ where k is a field with $d \in \mathbb{N}$, and set $\mathfrak{X} := (X_1, \ldots, X_d)R$.

Fact 2.7.1 ([2, Proposition 1.5.15])**.** Let $I \leq R$ be a monomial ideal. Then we have

$$\operatorname{depth}_R(\mathfrak{X}, R/I) = \operatorname{depth}_{R_{\mathfrak{X}}}(R_{\mathfrak{X}}/I_{\mathfrak{X}}).$$

Furthermore, the length of any maximal (homogeneous) (R/I)-sequence in $\mathfrak{X}$ is unique and is given by the depth of $R_{\mathfrak{X}}/I_{\mathfrak{X}}$.

Definition 2.7.2. Let $I \leq R$ be a monomial ideal. The *depth* of R/I is

$$\operatorname{depth}(R/I) = \operatorname{depth}_R(R/I) := \operatorname{depth}_{R_{\mathfrak{X}}}(R_{\mathfrak{X}}/I_{\mathfrak{X}}).$$

If $\dim(R/I) = \operatorname{depth}(R/I)$, then we say that R/I and I are *Cohen-Macaulay.*

Without Definition 2.7.2, the notation $\operatorname{depth}(R/I)$ is not well-defined, since $R = k[X_1, \ldots, X_d]$ is not local. For the same reason, we need to define the type of R/I over R, which we do next.

Definition 2.7.3. Let $I \leq R$ be a monomial ideal. The *type* of R/I is

$$\operatorname{type}_R(R/I) := \dim_k(\operatorname{Ext}^n_R(k, R/I))$$

where $n = \operatorname{depth}(R/I)$.

The following fact justifies Definition 2.7.3.

Fact 2.7.4. Let $I \leq R$ be a monomial ideal. Then

$$\operatorname{type}_R(R/I) = \operatorname{type}_{R_{\mathfrak{X}}}(R_{\mathfrak{X}}/I_{\mathfrak{X}}).$$

Remark. The polynomial ring $R = k[X_1, \ldots, X_d]$ is not local. However, we use $\mathfrak{X}$ in Definition 2.7.2 and 2.7.3 because $\mathfrak{X}$ is the unique maximal *homogeneous* ideal in R; so R is "local" with respect to homogeneous ideals.

We state some facts about dimension, depth, type, and regular sequences, then we give a proof that relates the type of R/I (where $I \leq R$ is a monomial ideal) and irredundant parametric decomposition of I.

Fact 2.7.5 ([2, Theorem 2.1.2])**.** Let $I \leq R$ be a monomial ideal and let $f_1, \ldots, f_r \in R$. If $f_1, \ldots, f_r$ is a homogeneous R/I-regular sequence, then

$$\operatorname{depth}(R/(I + (f_1, \ldots, f_r)R)) = \operatorname{depth}(R/I) - r$$

and

$$\dim(R/(I + (f_1, \ldots, f_r)R)) = \dim(R/I) - r.$$

If I is Cohen-Macaulay, then $f_1, \ldots, f_r$ is an R/I-regular sequence if and only if

$$\dim(R/(I + (f_1, \ldots, f_r)R)) = \dim(R/I) - r;$$

when these conditions hold, $R/(I + (f_1, \ldots, f_r)R)$ is Cohen-Macaulay. In particular, if I is Cohen-Macaulay and the sequence $f_1, \ldots, f_r$ is a maximal (R/I)-regular, then $\operatorname{depth}(R/I) = r$ and hence $\dim(R/(I + (f_1, \ldots, f_r)R)) = 0$.

Example 2.7.6. Let $R = k[X, Y, Z]$ and $I = \langle XYZ \rangle \leq R$. Fact 2.7.5 shows that the ideal I is Cohen-Macaulay with $\operatorname{depth}(R/I) = \dim(R/I) = 2$. (Alternatively, we can observe that I is an "odd open neighborhood ideal of a height 1 unmixed Δ-tree with 3 leaves X,Y, and Z", and apply Theorem 3.1.17.) We claim that $X - Y, X - Z \in \langle X, Y, Z \rangle$ is a maximal R/I-regular sequence. Indeed, since

$$\frac{R}{I + (X - Y, X - Z)R} \cong \frac{R}{\langle X^3 \rangle}$$

has dimension 0, Fact 2.7.5 implies that $X - Y, X - Z$ is a maximal R/I-regular sequence.

Fact 2.7.7 ([8, Proposition A.6.2])**.** Let $I \leq R$ be a monomial ideal such that R/I is Cohen-Macaulay, and let $f_1, \ldots, f_r \in R$ be a maximal homogeneous regular sequence for R/I. Then we have

$$\operatorname{type}(R/I) = \operatorname{type}(R/(I + (f_1, \ldots, f_r)R)).$$

Example 2.7.8. Let R and I be as in Example 2.7.6. Then we have

$$\operatorname{type}(R/I) = \operatorname{type}(R/(I + \langle X - Y, X - Z\rangle)) = \operatorname{type}\left(k[X]/\left\langle X^3\right\rangle\right).$$

Since $k[X]/\langle X^3\rangle$ has depth 0, we get

$$\operatorname{type}\left(k[X]/\left\langle X^3\right\rangle\right) = \dim_k\left(\operatorname{Ext}^0_{k[X]}\left(k, \frac{k[X]}{\langle X^3\rangle}\right)\right).$$

Fact 2.7.10 will help us compute the type of $k[X]/\langle X^3\rangle$ from the example above without computing the Ext functor explicitly. The following lemma will be used in the proof of Fact 2.7.10.

Lemma 2.7.9. *Let $I \leq R$ be an ideal that has a parametric decomposition $I = \bigcap_{i=1}^t Q_i$. Then we have depth$(R/I) = 0$.*

Proof. By definition of depth(R/I) it suffices to show that there is no R/I-regular element in $\mathfrak{X}$. Since Q_i is a parameter ideal for all $i \in [t]$, there exists $\underline{n}_i = (n_{i,1}, \ldots, n_{i,d}) \in \mathbb{N}_0^d$ for all $i \in [t]$ such that $Q_i = \left\langle X_1^{n_{i,1}}, \ldots, X_d^{n_{i,d}}\right\rangle$. Now let $f \in \mathfrak{X} \setminus I$ which is not a unit element. Then f is a polynomial in d variable with no constant term. Hence we can write

$$f = c_1\underline{X}^{\underline{a}_1} + \cdots + c_\ell\underline{X}^{\underline{a}_\ell}$$

where $\ell \in \mathbb{N}$, $\underline{a}_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{N}_0^d$ and $c_i \in k$ for all $i \in [t]$. We show that there is a monomial $\underline{X}^{\underline{m}} \in [[R]]$ such that $\overline{\underline{X}^{\underline{m}}} \neq 0$ and $f \cdot \overline{\underline{X}^{\underline{m}}} = 0$ in R/I. We prove this by inducting on ℓ.

Base case: $\ell = 1$. Then $f = c_1\underline{X}^{\underline{a}_1}$. Then there exists some index $j \in [d]$ such that $1 \le a_{1,j} < \max\{n_{1,j}, \ldots, n_{t,j}\}$; else $f \in I$ since the element $X_j^{\max\{n_{1,j},\ldots,n_{t,j}\}} \in Q_i$ for all $i \in [t]$ hence $f \in \bigcap_{i=1}^t Q_i = I$. Now set $m_j := \max\{n_{1,j}, \ldots, n_{t,j}\} - a_{1,j}$. Then the monomial $X_j^{m_j} \notin I$ since $\max\{n_{1,j}, \ldots, n_{t,j}\} - a_{1,j} < \max\{n_{1,j}, \ldots, n_{t,j}\}$ as $a_{i,j} \ge 1$. Thus $f \cdot X_j^{m_j} = c_1 X_1^{a_{1,1}} \cdots X_j^{\max\{n_{1,j},\ldots,n_{t,j}\}} \cdots X_d^{a_{1,d}} \in I$ so $f \cdot \overline{\underline{X}^{\underline{m}}} = 0$ in R/I.

Inductive case: $\ell > 1$. Consider $g = c_1\underline{X}^{\underline{a}_1} + \cdots + c_{\ell-1}\underline{X}^{\underline{a}_{\ell-1}}$. By our implied inductive hypothesis, there exists some monomial $\underline{X}^{\underline{m}} \in [[R]]$ with $\underline{m} = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$ so that $g \cdot \overline{\underline{X}^{\underline{m}}} = 0 \neq \overline{\underline{X}^{\underline{m}}}$ in R/I. By the same argument in the base case, there is some index $j \in [d]$ so that $1 \le a_{\ell,j} < \max\{n_{1,j}, \ldots, n_{t,j}\}$. Now set $m'_j := \max\{m_j, \max\{n_{1,j}, \ldots, n_{t,j}\} - a_{\ell,j}\}$ and $\underline{m}' = (m_1, \ldots, m_{j-1}, m'_j, m_{j+1}, \ldots, m_d) \in \mathbb{N}_0^d$. Note that $\overline{\underline{X}^{\underline{m}'}} \neq 0$ since $\overline{\underline{X}^{\underline{m}}} \neq 0$ and $m'_j < \max\{n_{1,j}, \ldots, n_{t,j}\}$. Hence we get

$$f \cdot \underline{X}^{\underline{m}'} = (g + c_\ell \underline{X}^{\underline{a}_\ell}) \cdot \underline{X}^{\underline{m}'} = g \cdot \underline{X}^{\underline{m}'} + c_\ell \underline{X}^{\underline{a}_k} \cdot \underline{X}^{\underline{m}'} \in I$$

since $\underline{X}^{\underline{m}} \cdot g \in I$ with $\underline{X}^{\underline{m}} | \underline{X}^{\underline{m}'}$ and $m'_j + a_{k,j} \ge \max\{n_{1,j}, \ldots, n_{t,j}\}$ hence $X_j^{m'_j + a_{\ell,j}} \in I$. Thus $f \cdot \overline{\underline{X}^{\underline{m}'}} = 0$ in R/I.

Thus there is no R/I -regular element in $\mathfrak{X}$, therefore the length of a maximal R/I-regular sequence in $\mathfrak{X}$ is 0. $\square$

Remark. Another proof of Lemma 2.7.9 can be given once we know that depth is bounded above by dimension. Indeed, since I has a parametric decomposition, every ideal in the decomposition has d generators, so Fact 2.6.15 implies $0 \le \operatorname{depth}(R/I) \le \dim_R(R/I) = 0$.

Fact 2.7.10. Let $I \leq R$ be a monomial ideal with irredundant parametric decomposition $I = \bigcap_{i=1}^{t} P_R(f_i)$ where $f_i \in [\![R]\!]$ for all $i \in [t]$. Then we have $\operatorname{type}(R/I) = t$.

Proof. By Fact 2.6.22 and the uniqueness of irredundant m-irreducible decompositions, the monomials $f_1, \ldots, f_t$ are the distinct I-corner elements in R, and we get $|C_R(I)| = t$. Then we have

$$\begin{aligned} \operatorname{type}(R/I) &= \dim_k(\operatorname{Ext}_R^0(k, R/I)) && \text{(Lemma 2.7.9)} \\ &= \dim_k(\operatorname{Hom}_R(k, R/I)). && \text{(Fact 2.3.9)} \end{aligned}$$

Hence it suffices to show that $|C_R(I)| = \dim_k(\operatorname{Hom}_R(k, R/I))$. By Proposition 2.6.28, the k-vector space $(I :_R \mathfrak{X})/I$ has $\overline{f_1}, \ldots, \overline{f_t}$ as a basis. For each $i \in [t]$, define the map $\mu_i : R/\mathfrak{X} \to R/I$ via $\mu_i(\overline{r}) = \overline{rf_i}$. We show that the μ_i's are well-defined. Fix some $i \in [t]$, and let $r_1, r_2 \in R$ such that $\overline{r_1} = \overline{r_2}$ in $R/\mathfrak{X}$. Then $r_1 - r_2 \in \mathfrak{X}$, so we get $(r_1 - r_2)f_i \in I$ since $f_i \in C_R(I)$. Thus

$$\mu_i(r_1) - \mu_i(r_2) = \overline{r_1 f_i} - \overline{r_2 f_i} = \overline{r_1 f_i - r_2 f_i} = \overline{(r_1 - r_2) f_i} = \overline{0}$$

which gives $\mu_i(r_1) = \mu_i(r_2)$. With this, it is straightforward to show that the μ_i's are in $\operatorname{Hom}_R(R/\mathfrak{X}, R/I)$. Now set $\overline{C_R(I)} = \{\overline{f_1}, \ldots, \overline{f_t}\} \subseteq (I :_k \mathfrak{X})/I$, and define a map $\Phi : \overline{C_R(I)} \to \operatorname{Hom}_R(A, R/I)$ by $\Phi(f_i) = \mu_i$. Since $\overline{C_R(I)} \subseteq (I :_R \mathfrak{X})/I$ and $\operatorname{Hom}_R(A, R/I)$ are finite dimensional k-vector spaces (by [12, Remark IX.3.4]), by the universal mapping property for k-vector spaces, the map Φ induces a k-linear map $\hat{\Phi}$

such that

$$\hat{\Phi} : (I :_R \mathfrak{X})/I \longrightarrow \mathrm{Hom}_R(k, R/I)$$

$$\overline{f_i} \longmapsto \mu_i \ , \quad \forall i \in [t]$$

$$\sum_{i=1}^{t} \overline{a_i}\overline{f_i} \longmapsto \sum_{i=1}^{t} \overline{a_i}\mu_i.$$

Now we show that $\hat{\Phi}$ is a bijection.

Let $x = \sum_{i=1}^{t} \overline{a_i}\overline{f_i} \in (I :_R \mathfrak{X})/I$ such that $x \in \ker(\hat{\Phi})$. Then we have

$$\overline{0} = (\hat{\Phi}(x))(1) = \left(\hat{\Phi}\left(\sum_{i=1}^{t} a_i\overline{f_i}\right)\right)(1) = \left(\sum_{i=1}^{t} a_i\mu_i\right)(1) = \sum_{i=1}^{t} a_i\mu_i(1) = \sum_{i=1}^{t} a_i\overline{f_i}.$$

Since the $\overline{f_i}$'s are linearly independent over A, we must have $a_1 = \cdots = a_t = 0$. Thus $x = \overline{0}$ so $\hat{\Phi}$ is injective.

Let $\psi \in \mathrm{Hom}_R(R/\mathfrak{X}, R/I)$. If $\psi = 0$, then $\psi = \hat{\Phi}(0)$ since $\hat{\Phi}$ is linear. So suppose that $\psi \neq 0$. Let $s \in R$ be such that $\psi(1+\mathfrak{X}) = s + I$. We claim that $s\mathfrak{X} \subseteq I$. Assume not, then there is some index $i \in [d]$ such that $X_i s \notin I$. Then we have

$$0 + I = \phi(0 + \mathfrak{X}) = \phi(X_i + \mathfrak{X}) = X_i s + I \neq 0 + I$$

a contradiction. Thus, we have $s\mathfrak{X} \subseteq I$. Hence we have $s \in (I :_R \mathfrak{X}) = I + (C_R(I))R = I + (f_1, \ldots, f_t)R$ by definition of colon ideals and Lemma 2.6.27. So, there are $a_1, \ldots, a_t \in R$ such that $\overline{s} = \sum_{i=1}^{t} \overline{a_i}(f_i + I)$. For any $r \in R$ we have

$$\begin{aligned}\psi(r + \mathfrak{X}) = rs + I = r\left(\sum_{i=1}^{t} a_i(f_i + I)\right) = r\left(\sum_{i=1}^{t} a_i\mu_i(1 + \mathfrak{X})\right)\\ = \sum_{i=1}^{t} ra_i\mu_i(1 + \mathfrak{X}) = \sum_{i=1}^{t} a_i\mu_i(r + \mathfrak{X}) = \left(\sum_{i=1}^{t} a_i\mu_i\right)(r + \mathfrak{X})\end{aligned}$$

$$= \hat{\Phi}\left(\sum_{i=1}^{t} \overline{a_i}\overline{f_i}\right)(r + \mathfrak{X}).$$

Therefore, $\hat{\Phi}\left(\sum_{i=1}^{t} \overline{a_i}\overline{f_i}\right) = \psi$ so $\hat{\Phi}$ is surjective. □

Example 2.7.11. Let R and I be as in Example 2.7.6. By Example 2.7.8, we get

$$\operatorname{type}(R/I) = \operatorname{type}(k[X]/\left\langle X^3\right\rangle).$$

Since $\langle X^3\rangle$ is a parameter ideal in $k[X]$, the ideal $\langle X^3\rangle$ is its own irredundant parametric decomposition (with a single ideal in the intersection). Thus by Fact 2.7.10, we get

$$\operatorname{type}(R/I) = \operatorname{type}(k[X]/\left\langle X^3\right\rangle) = 1.$$

2.8 Total Dominating Sets and Open Neighborhood Ideals

This section contains some graph theoretic results that are related to the main result of the thesis.

Assumptions. For the entire section, assume that G is a finite simple graph and k is a field. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. If the graph we consider is clear from context, then we write $V := V(G)$ and $E := E(G)$. Since the graph is simple, every edge $e \in E$ will be denoted as uv (or vu) where $u, v \in V$ are the vertices incident to e. Denote $k[V(G)] = k[V]$ by the polynomial ring over k whose variables are the vertices of G.

In order to define the main algebraic object related to graphs in this thesis, we collect a few more graph theoretic definitions.

Definition 2.8.1. For $v \in V$, consider the *open neighborhood* of v in G,

$$N_G(v) := \{u \in V : uv \in E(G)\}$$

and for $V' \subseteq V$, we set

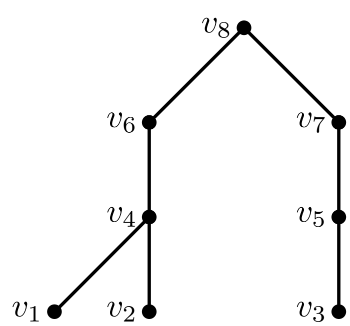
$$N_G(V') := \bigcup_{v \in V'} N_G(v).$$

For $U \subseteq V$, define the *monomial of* U in $k[V]$ by

$$X_U := \prod_{v \in U} v, \quad \text{and define} \quad P_U := \langle \{v : v \in U\} \rangle .$$

If the graph G is clear in the context, then we may drop the G in the subscript of N.

Example 2.8.2. Let T be the following graph with vertex set $\{v_1, v_2, \ldots, v_8\}$:



Then $k[V] = k[v_1, \ldots, v_8]$. We have $N_T(v_4) = \{v_1, v_2, v_6\}$, $N_T(v_8) = \{v_6, v_7\}$, and $N_T(\{v_4, v_5, v_8\}) = \{v_1, v_2, v_3, v_6, v_7\}$. Setting $U = N_T(v_4)$, we get $X_U = v_1v_2v_6$ and $P_U = \langle v_1, v_2, v_6 \rangle$ in R.

Definition 2.8.3. The *open neighborhood ideal of* G in $k[V]$ is the ideal generated by the monomials of the open neighborhoods of the vertices of G:

$$N(G) := \left(\left\{X_{N_G(v)} \mid v \in V\right\}\right) R = \left\langle \left\{X_{N_G(v)} \mid v \in V\right\} \right\rangle .$$

Example 2.8.4. Consider the graph T and the ring R in Example 2.8.2. Then we have

$$\begin{aligned} N(T) &= \left(\left\{X_{N_T(v)} : v \in V\right\}\right) R \\ &= \left(X_{N_T(v_1)}, \dots, X_{N_T(v_8)}\right) R \\ &= (v_4, v_5, v_1v_2v_6, v_3v_7, v_4v_8, v_5v_8, v_6v_5)\, R \\ &= (v_4, v_5, v_1v_2v_6, v_3v_7, v_6v_5)\, R. \end{aligned}$$

Remark. By Definition 2.8.1, open neighborhood ideals are always monomial ideals generated by 'square-free' monomials; a monomial $f \in k[X_1, \dots, X_d]$ is *square-free* if $f = \prod_{i=1}^{n} X_i^{\varepsilon_i}$ where $\varepsilon_i \in \{0, 1\}$ for all $i \in [d]$.

Next, we define some special type of trees that will be used to characterize 'unmixed trees' with respect to 'total domination.'

Definition 2.8.5. Let C be a finite set of colors. A *graph coloring* (or simply *coloring*) of G is a function $\chi : V \to C$ such that for all $uv \in E$, we have $\chi(u) \neq \chi(v)$; that is, no two adjacent vertices share the same color. If $|C| = n \in \mathbb{N}$, then we say χ is an *n-coloring* of G. A coloring χ is a *minimal graph coloring* of G if n is the minimum cardinality of C such that a coloring exists.

Fact 2.8.6. It is straightforward to show that every bipartite graph has a 2-coloring. Hence, every tree has a 2-coloring.

Example 2.8.7. Consider the graph T from Example 2.8.2. A minimal coloring χ with $C = \{\text{blue}, \text{red}\}$ is the graph in Figure 2.2 below. In this case, we have $\chi(\{v_1, v_2, v_3, v_6, v_7\}) = \{\text{blue}\}$ and $\chi(\{v_4, v_5, v_8\}) = \{\text{red}\}$. Hence we have $\chi^{-1}(\text{blue}) = \{v_1, v_2, v_3, v_6, v_7\}$ and $\chi^{-1}(\text{red}) = \{v_4, v_5, v_8\}$.

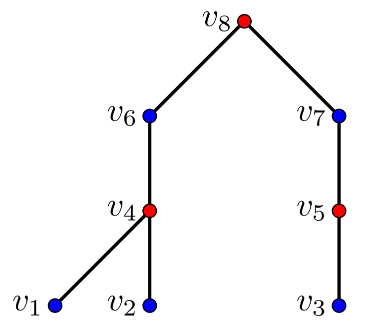


Figure 2.2: A 2-coloring of T from Example 2.8.2

Assumptions. Since we only consider trees, we always assume that χ is a 2-coloring with the set of colors $C = \{\text{blue}, \text{red}\} =: \{B, R\}$.

Definition 2.8.8. Let $n \in \mathbb{N}$. For $v_0, v_n \in V$, a *path* from v_0 to v_n in G is a sequence of distinct vertices $P := v_0 v_1 \cdots v_n$ such that $v_{i-1}v_i \in E$ for all $i \in [n]$. The value $l(P) := n$ is the *length* of P. For $u, v \in V$, the *distance* between u and v is

$$d_G(u,v) := \min \{l(P) : P \text{ is a path from } u \text{ to } v\}.$$

Definition 2.8.9. Let T be a finite simple graph. The *degree* of $v \in V$, denoted by $\deg_T(v)$, is the number of vertices v is adjacent to in T. A vertex $\ell \in V$ is a *leaf* if $\deg(\ell) = 1$, and every vertex adjacent to ℓ is called a *support vertex*. Suppose that T is a tree. Set $\text{height}_T(\ell) = 0$ for all leaves $\ell \in V$. For each non-leaf $v \in V$, the *height of v* is

$$\text{height}_T(v) := \min \{d_T(v,\ell) : \ell \in V, \deg(\ell) = 1\}.$$

The *height of T* is

$$\text{height}(T) := \max \{\text{height}_T(v) : v \in V\}.$$

For $t \in \mathbb{N}_0$, we set

$$V_t = V_t(T) := \{v \in V : \text{height}_T(v) = t\}\,.$$

For isolated vertices, i.e., vertices $v \in V$ with $\deg(v) = 0$, we set $\text{height}(v) = 0$. Hence if T is a graph with a single vertex, then set $\text{height}(T) = 0$.

Example 2.8.10. Consider the tree T from Example 2.8.7. The set of all leaves in T is $V_0(T) = \{v_1, v_2, v_3\}$, the set of support vertices is $V_1(T) = \{v_4, v_5\}$. Additionally, we have $V_2(T) = \{v_6, v_7\}$ and $V_3(T) = \{v_8\}$. Thus the height of T is $\text{height}(T) = 3$.

Now we are ready to define the 'Δ-trees.'

Definition 2.8.11. A tree T is a Δ-*tree* if no two vertices of the same height are adjacent.

Fact 2.8.12 ([7, Proposition 4.2.23])**.** Let T be a tree with a 2-coloring. The followings are equivalent:

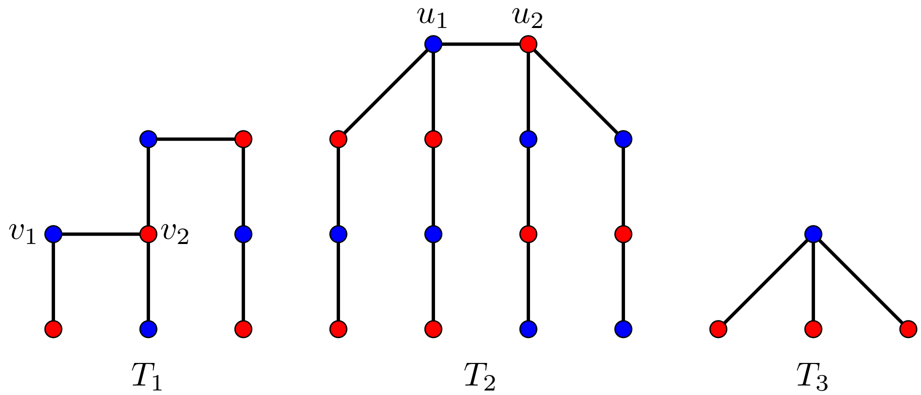
(a) T is a Δ-tree.

(b) Any two vertices of the same height have the same color.

(c) Every leaf has the same color (hence $|\chi(V_0)| = 1$).

Example 2.8.13. Consider the following trees with 2-coloring:

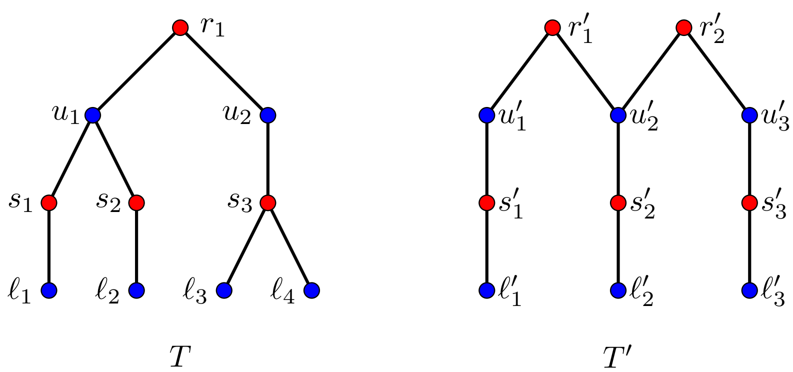
The trees T_1 and T_2 are not Δ-trees since $v_1, v_2 \in V(T_1)$ have height 1 but are adjacent, and $u_1, u_2 \in V(T_2)$ have height 3 but are adjacent. Also, using Fact 2.8.12, T_1 and T_2 are not Δ-trees since the leaves in each tree do not have the same color. On the other hand, T_3 and the tree from Example 2.8.7 are both Δ-trees by definition, or since all of their leaves share the same color.



Now we define 'total dominating sets' and 'unmixed graphs.' The 'total dominating sets' of G and the open neighborhood ideal of G have a close connection which will be stated in Fact 2.8.16.

Definition 2.8.14. Let $D \subseteq V$. Then D is a *total dominating set* of G (or *TD-set* of G, for short) if $N_G(D) = V$. We say D is *minimal* if there does not exists any proper subset $D' \subsetneq D$ such that $N(D') = V$, i.e., if D does not properly contain another TD-set of G. The graph G is *unmixed* (with respect to TD-sets) if every minimal TD-set of G has the same size. If G is a graph with a single vertex, then we declare the empty set $\emptyset$ to be its unique minimal TD-set.

Example 2.8.15. Consider the following Δ-trees T and T':



A TD-set of T is $S := \{s_1, s_2, s_3, u_1, u_2, \ell_3\}$. But S is not minimal since the proper subset $\{s_1, s_2, s_3, u_1, \ell_3\} =: S' \subsetneq S$ is also a TD-set. Moreover, the set S' is a minimal TD-set of T since no proper subset of S' is a TD-set. The tree T is mixed since $S'' := \{s_1, s_2, s_3, \ell_1, \ell_2, u_3\}$ is also a minimal TD-set of T with size 6, but S' is a minimal TD-set of T of size 5.

On the other hand, the minimal TD-sets of T' are:

$$V_1(T') \cup \{u'_1, u'_2, u'_3\} \qquad V_1(T') \cup \{u'_1, \ell'_2, u'_3\} \qquad V_1(T') \cup \{\ell'_1, u'_2, \ell'_3\}$$
$$V_1(T') \cup \{u'_1 u'_2, \ell'_3\} \qquad V_1(T') \cup \{\ell'_1, u'_2, u'_3\}$$

where $V_1(T') = \{s'_1, s'_2, s'_3\}$. Since all minimal TD-sets of T' have size 6, T' is unmixed.

Fact 2.8.16 ([7, Theorem 4.1.5])**.** The open neighborhood ideal of G in R has the following m-irreducible decomposition

$$N(G) = \bigcap_{V'} P_{V'} = \bigcap_{V' \; min} P_{V'}$$

where the first intersection is taken over all TD-sets of G, and the second intersection is taken over all minimal TD-sets of G. Moreover, the second decomposition is irredundant.

Example 2.8.17. We demonstrate Fact 2.8.16 for the tree T' from the previous example. With $V_1 := V_1(T')$, set

$$D_1 := V_1 \cup \{u'_1, u'_2, u'_3\} \qquad D_2 := V_1 \cup \{u'_1, \ell'_2, u'_3\} \qquad D_3 := V_1 \cup \{\ell'_1, u'_2, \ell'_3\}$$
$$D_4 := V_1 \cup \{u'_1 u'_2, \ell'_3\} \qquad D_5 := V_1 \cup \{\ell'_1, u'_2, u'_3\}$$

Using Fact 2.6.13 we have

$$
\begin{aligned}
N(T') &= \left\langle s_1',\ s_2',\ s_3',\ \ell_1'u_1',\ \ell_2'u_2',\ \ell_3'u_3',\ \cancel{s_1'r_1'},\ \cancel{r_1'r_2's_2'},\ \cancel{r_2's_3'},\ u_1'u_2',\ u_2'u_3' \right\rangle \\
&= \langle s_1',\ s_2',\ s_3',\ \ell_1'u_1',\ \ell_2'u_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&= \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2'u_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2'u_2',\ \ell_3'u_3',\ \cancel{u_1'u_2'},\ u_2'u_3' \rangle \\
&= \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2'u_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2'u_2',\ \ell_3'u_3',\ u_2'u_3' \rangle \\
&= \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ u_2',\ \ell_3'u_3',\ \cancel{u_1'u_2'},\ \cancel{u_2'u_3'} \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3'u_3',\ u_2'u_3' \rangle \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ u_2',\ \ell_3'u_3',\ \cancel{u_2'u_3'} \rangle \\
&= \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ u_2',\ \ell_3'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3'u_3',\ u_2'u_3' \rangle \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ u_2',\ \ell_3'u_3' \rangle \\
&= \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \cap P_{D_3} \cap P_{D_5} \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3'u_3',\ u_2'u_3' \rangle \cap P_{D_1} \cap P_{D_4} \\
&= P_{D_1} \cap P_{D_3} \cap P_{D_4} \cap P_{D_5} \cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3'u_3',\ u_2'u_3' \rangle \\
&= P_{D_1} \cap P_{D_3} \cap P_{D_4} \cap P_{D_5} \cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3',\ u_2'u_3' \rangle \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ u_3',\ \cancel{u_2'u_3'} \rangle \\
&= P_{D_1} \cap P_{D_3} \cap P_{D_4} \cap P_{D_5} \cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3',\ u_2'u_3' \rangle \cap P_{D_2} \\
&= \left(\bigcap_{i=1}^{5} P_{D_i} \right) \cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_1'u_2',\ u_2'u_3' \rangle \\
&\quad \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3',\ u_2'u_3' \rangle
\end{aligned}
$$

$$= \left(\bigcap_{i=1}^{5} P_{D_i}\right) \cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ \cancel{u_1'u_2'},\ u_2' \rangle$$

$$\cap \langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \cancel{\ell_3'u_3'},\ u_1'u_2',\ u_3' \rangle \cap \langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3',\ u_2'u_3' \rangle$$

$$= \bigcap_{i=1}^{5} P_{D_i}$$

where the last equality holds since

$$\langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ \ell_3'u_3',\ u_2' \rangle \supseteq P_{D_3} \cap P_{D_5}$$
$$\langle s_1',\ s_2',\ s_3',\ \ell_1',\ \ell_2',\ u_1'u_2',\ u_3' \rangle \supseteq P_{D_2} \cap P_{D_5}$$
$$\langle s_1',\ s_2',\ s_3',\ u_1',\ \ell_2',\ \ell_3',\ u_2'u_3' \rangle \supseteq P_{D_2} \cap P_{D_4}.$$

At the end of this section, we provide a way to "decompose" any tree into two Δ-trees with a given 2-coloring. First, we describe some useful properties of Δ-trees.

Fact 2.8.18 ([7, Theorem 4.2.36])**.** Let T be a Δ-tree. Then T is unmixed if and only if

(a) $\text{height}(T) \leq 3$,

(b) $\forall \ell_1, \ell_2 \in V_0$, $d_T(\ell_1, \ell_2) \neq 4$, and

(c) $\forall v \in V_1$, $|N_T(v) \cap V_2| \leq 1$.

Example 2.8.19. We verify Fact 2.8.18 for the trees from Example 2.8.15, resketched in Figure 2.3 below. The tree T' satisfies all the conditions in Fact 2.8.18 and indeed we saw in 2.8.15 that T' is unmixed. On the other hand, the tree T fails to satisfy condition (b) since $d_T(\ell_1, \ell_2) = 4$ and indeed we saw in 2.8.15 that T is mixed.

The unmixedness condition for Δ-trees in Fact 2.8.18 can be restated using the following lemma.

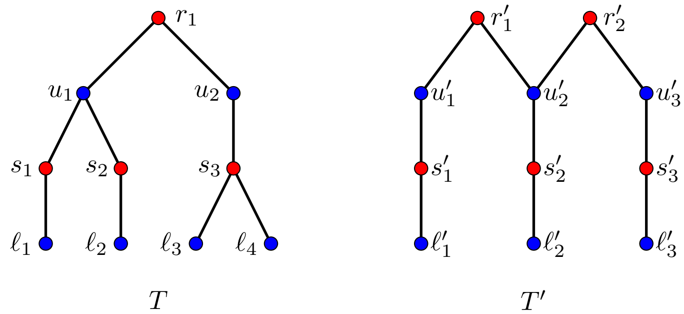


Figure 2.3: Trees from Example 2.8.15

Lemma 2.8.20. *Let T be a Δ-tree. Then for all leaves $\ell_1, \ell_2 \in V_0$, $d_T(\ell_1, \ell_2) \neq 4$ if and only if for all $v \in V_2$, we have $|N(v) \cap V_1| = 1$.*

Proof. ($\Rightarrow$) Assume by way of contradiction that there is a vertex $v \in V_2$ such that $|N(v) \cap V_1| \neq 1$. By definition of height, we must have $|N_T(v) \cap V_1| \geq 1$. So, there are distinct $s_1, s_2 \in N(v) \cap V_1$. Then there are leaves $\ell_1, \ell_2 \in V_0$ such that $\ell_1 s_1, \ell_2 s_2 \in E$. Thus this forms a path between ℓ_1 and ℓ_2, namely $\ell_1 s_1 v s_2 \ell_2$, contradicting the assumption that there is no path of length 4 between any two leaves.

($\Leftarrow$) Let $\ell_1, \ell_2 \in V_0$ be two leaves. Then there are support vertices $s_1, s_2 \in V_1$ such that $\ell_1 s_1, \ell_2 s_2 \in E$. If $s_1 = s_2$, then $\ell_1 s_1 \ell_2$ forms a path of length 2 between ℓ_1 and ℓ_2. Thus by the uniqueness of path between two vertices in a tree, we get $d_T(\ell_1, \ell_2) = 2 \neq 4$. Now suppose that $s_1 \neq s_2$. Since T is a Δ-tree, there is no edge between s_1 and s_2. Thus we must have $d_T(\ell_1, \ell_2) > 3$. Assume by way of contradiction that $d_T(\ell_1, \ell_2) = 4$. Then there exists $v \in V_2$ such that $\ell_1 s_1 v s_2 \ell_2$ forms a path of length 4. This implies that $s_1, s_2 \in N(v)$, so $|N(v) \cap V_1| \geq 2$, contradicting the assumption that $|N(v) \cap V_1| = 1$ as $v \in V_2$. Thus $d_T(\ell_1, \ell_2) \neq 4$. $\square$

Now we state a new characterization of unmixed Δ-trees.

Theorem 2.8.21. *Let T be a Δ-tree. Then T is unmixed if and only if*

(a) height$(T) \leq 3$,

(b) $\forall v \in V_2$, $|N_T(v) \cap V_1| = 1$,

(c) $\forall v \in V_1$, $|N_T(v) \cap V_2| \leq 1$.

Proof. Apply Lemma 2.8.20 to condition (b) in Fact 2.8.18. □

Remark 2.8.22. Given an unmixed Δ-tree T of height 3, condition (c) in Theorem 2.8.21 becomes an equality. If the conditions (b) and (c) in Theorem 2.8.21 hold for T, then the set $\{uv \in E : u \in V_1, v \in V_2\}$ describes a bijection between V_1 and V_2 by the pigeonhole principle. Also, the inequality in condition (c) becomes an equality in this setting.

Example 2.8.23. Here we give some intuition of unmixed Δ-trees with respect to their heights. Let T be an unmixed Δ-tree. If $\text{height}(T) = 0$, then T must be the graph with a single isolated vertex. If $\text{height}(T) = 1$, then Theorem 2.8.21 shows that T must be a "star graph" shown in Figure 2.4, i.e., T has a unique height 1 vertex s_1, and has n_0 leaves with $n_0 \geq 2$. (If $n_0 = 1$, then T is a path of 2 vertices which is not a Δ-tree since both vertices are leaves).

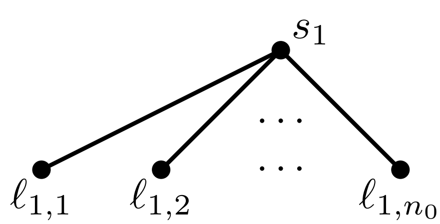


Figure 2.4: Unmixed Tree of height 1, "Star graph".

There is no height 2 unmixed Δ-tree. Indeed, let T' be a Δ-tree of height-2, and let $u \in V_2(T')$. If $\deg(u) = 1$, then u becomes a leaf, contradicting our assumption that $\text{height}(u) = 2$. So, $\deg(u) \geq 2$. Since T' is a Δ-tree of height 2, we have $N(u) \subseteq V_1(T')$. Hence $|N(u) \cap V_1(T')| \geq 2$, violating condition (b) in Theorem 2.8.21. Hence the only unmixed Δ-trees have heights 0,1, and 3.

Next, we give a way to "decompose" any given tree into two "Δ-forests;" see Fact 2.8.28 below.

Definition 2.8.24. Let $S \subseteq V$. The *subgraph of G induced by S*, denoted G_S, is the subgraph of G such that $V(G_S) = S$ and $E(G_S) = \{uv : u, v \in S, uv \in E(G)\}$.

Definition 2.8.25. Let T be a tree with a 2-coloring $\chi : V(T) \to \{B, R\}$. We define the *blue interior graph* of T to be the subgraph T_B of T induced by the set

$$V \setminus N[V_1 \cap \chi^{-1}(B)]$$

where $N[S] := N(S) \cup S$ is the closed neighborhood of S for any $S \subseteq V$; i.e., T_B is the subgraph of T induced by deleting all blue support vertices together with their neighbors. Similarly, we define T_R to be the *red interior graph* of T induced by the set

$$V \setminus N[V_1 \cap \chi^{-1}(R)].$$

Example 2.8.26. Figure 2.5 shows a 2-colored tree T, and Figure 2.6 below shows its blue and red interior graphs.

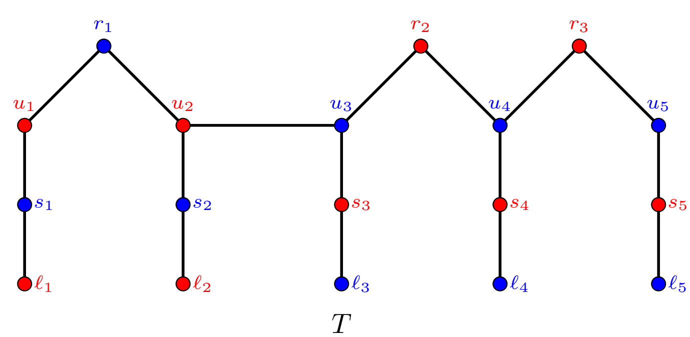


Figure 2.5: A 2-coloring of a tree T

This example shows that the interior graphs can be forests.

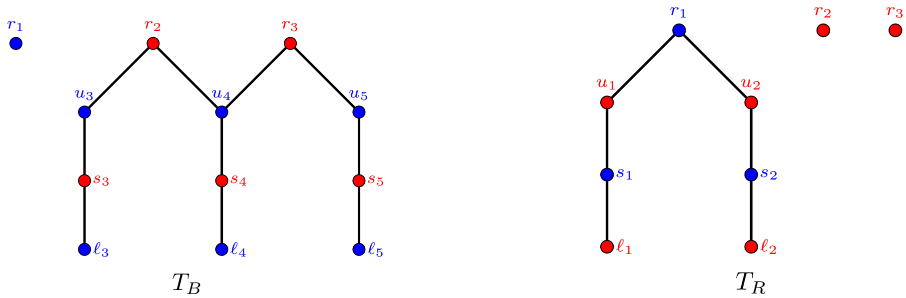


Figure 2.6: Interior graphs of T from Figure 2.5

Example 2.8.27. Consider the tree T in Figure 2.7. Its interior graphs T_B and T_R are shown in Figure 2.8. For this specific tree T, its interior graphs are isomorphic.

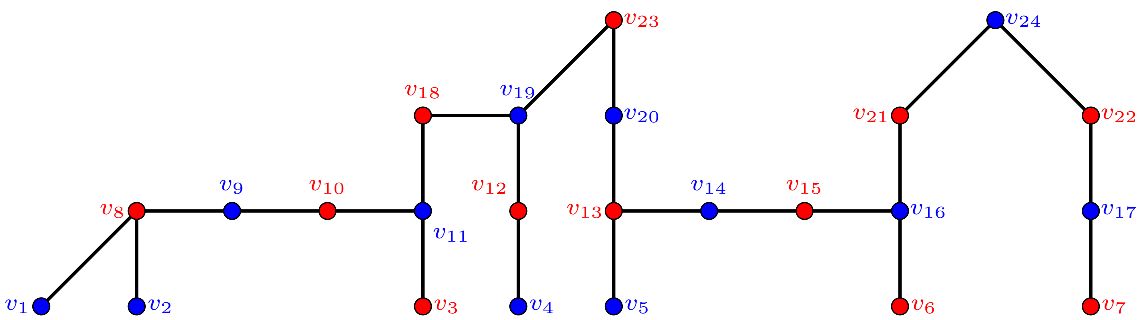


Figure 2.7: A tree T

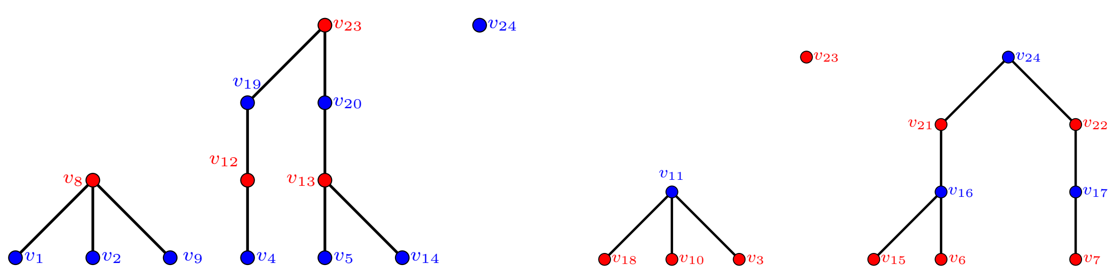


Figure 2.8: T_B (left) and T_R (right) of T in Figure 2.7

All connected components of the interior graphs in Examples 2.8.27 and 2.8.26 are Δ-trees. This is true in general and is proved in the following fact.

Fact 2.8.28. Every connected component of T_B and T_R is a Δ-tree. Hence we say that T_B and T_R are "Δ-*forests.*"

Proof. Without loss of generality, consider the blue interior graph T_B. By Definition 2.8.25, we delete all blue support vertices with their neighboring vertices (that are red colored) in T when we construct T_B. Hence the leaves and the isolated vertices in T_B must be blue color, implying that the connected components of T_B must be Δ-trees by Fact 2.8.12. □

Now we state the unmixedness condition for a general tree.

Fact 2.8.29 ([7, Theorem 4.2.37])**.** Let T be a tree with a given 2-coloring. Then T is unmixed if and only if both T_B and T_R are unmixed.

Example 2.8.30. Let T be the tree in Figure 2.5. Since its interior graphs T_B and T_R in Figure 2.6 are unmixed, T is unmixed by Fact 2.8.29.

2.9 Simplicial Complexes and Monomial Ideals

In this section, we introduce the basic notions used in Stanley-Reisner Theory.

Assumptions. We set $V = \{v_1, \ldots, v_d\}$ with $d \in \mathbb{N}$ and let k be a field.

Definition 2.9.1. A *simplicial complex* on V is a non-empty collection $\Delta \subseteq 2^V$ such that for any $F, G \subseteq V$, if $F \subseteq G$ and $G \in \Delta$, then $F \in \Delta$; i.e., Δ is closed under subsets. Each element of Δ is called a *face* of Δ. A maximal element of Δ with respect to containment is a *facet* of Δ. We say Δ is *pure* if every facet of Δ has the same size.

Notation 2.9.2. Let Δ be a simplicial complex with facets $\mathcal{F}_1, \ldots, \mathcal{F}_n$. Since simplicial complexes are closed under taking subsets, to describe a simplicial complex it is sufficient to list the facets. In this case, we write $\Delta =: \langle \mathcal{F}_1, \ldots, \mathcal{F}_n \rangle$.

Example 2.9.3. For each $n \in \mathbb{N}$, the power set $2^{[n]}$ is a simplicial complex with unique facet $[n]$. For instance, the simplicial complexes $2^{[3]}$ and $2^{[4]}$ can be viewed as a solid triangle and a solid tetrahedron, respectively, as in Figure 2.9.

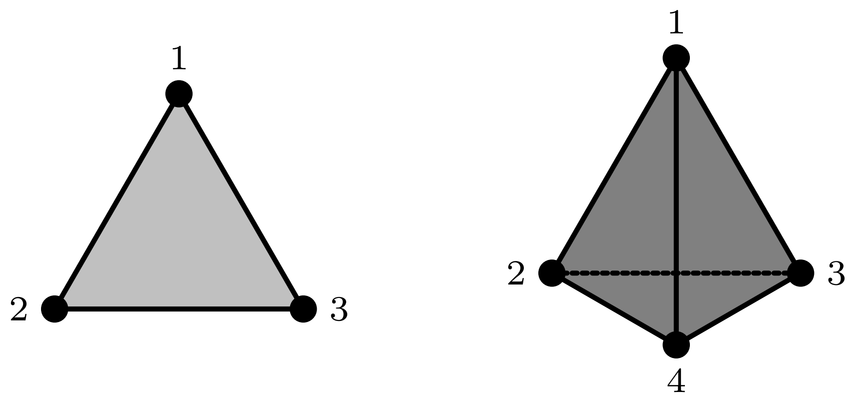


Figure 2.9: Geometric realization of $2^{[3]}$ and $2^{[4]}$

Now set $V := \{x_1, x_2, x_3, x_4\}$, and let

$$\Delta := \{\{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_3, x_4\} \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \emptyset\}.$$

One checks readily that the set Δ is a simplicial complex on V. Since $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, and $\{x_2, x_3, x_4\}$ are the facets of Δ, we get

$$\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_2, x_3, x_4\} \rangle .$$

Figure 2.10 shown below is a geometric realization of Δ, as three shaded triangles $\{x_1, x_2, x_3\}$, $\{x_1, x_2, x_4\}$, and $\{x_2, x_3, x_4\}$ glued together along edges.

One can use simplicial complexes to construct square-free monomial ideals as in the following definition.

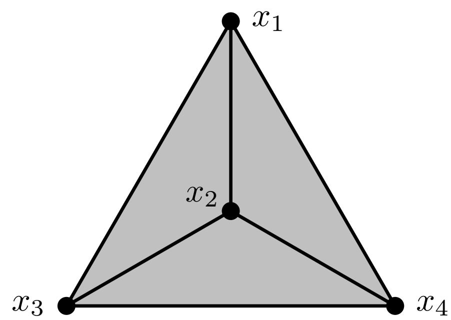


Figure 2.10: Simplicial complex Δ

Definition 2.9.4. Let Δ be a simplicial complex on V, and set $R = k[V]$. The *Stanley-Reisner ideal* of R associated to Δ is the ideal "generated by the non-faces of Δ:"

$$J_\Delta := (\{X_U : U \subseteq V \text{ and } U \notin \Delta\})\, R$$

where X_U is the monomial in R given in Definition 2.8.1. The quotient ring

$$A[\Delta] := R/J_\Delta$$

is called the *Stanley-Reisner ring* of Δ over A.

Example 2.9.5. Let Δ be the simplicial complex from Example 2.9.3. Then

$$J_\Delta = \langle x_1x_2x_3x_4, x_1x_3x_4\rangle = \langle x_1x_3x_4\rangle\,.$$

From the second equality, we see that the generators of J_Δ correspond to the "minimal non-faces of Δ."

The Stanley-Reisner ideal has the following m-irreducible decomposition.

Fact 2.9.6 ([10, Theorem 4.5.4])**.** Let Δ be a simplicial complex on $V = \{v_1, \ldots, v_d\}$ and set $R = k[V]$. Let $\mathcal{F}$ be the set of facets of Δ. The Stanley-Reisner ideal $J_\Delta \subseteq R$

has the following irredundant m-irreducible decomposition

$$J_\Delta = \bigcap_{F \in \mathcal{F}} P_{V \setminus F}.$$

One important property of simplicial complex is 'shellability.' Determining whether a given pure simplical complex is shellable is known to be an NP-complete problem [6]. For us, shellability is used as a tool to determine if a Stanley-Reisner ring of a given simplicial complex is Cohen-Macaulay (see Theorem 2.9.10).

Definition 2.9.7 ([2, Definition 5.1.11])**.** Let Δ be a pure simplicial complex on V. We say Δ is *shellable* if one of the following equivalent conditions is satisfied: the facets of Δ can be totally ordered, say $F_1, \ldots, F_m$, so that

(a) $\langle F_i \rangle \cap \langle F_1, \ldots, F_{i-1} \rangle$ is generated by a non-empty set of maximal proper faces of $\langle F_i \rangle$ for all $2 \leq i \leq m$, or

(b) for all i, j with $1 \leq j < i \leq m$, there exist some $v \in F_i \setminus F_j$ and some $k \in [i-1]$ with $F_i \setminus F_k = \{v\}$.

Any total ordering satisfying the conditions above is called a *shelling* of Δ.

Example 2.9.8. Let Δ be the simplicial complex from Example 2.9.3. Set $F_1 := \{x_1, x_2, x_3\}$, $F_2 := \{x_1, x_2, x_4\}$, and $F_3 := \{x_2, x_3, x_4\}$. We show that Δ is shellable using condition (b) in Definition 2.9.7 with the total order on the facets given by their indexing:

Case 1: $j = 1$, $i = 2$. Then $\{x_4\} = F_2 \setminus F_1$ hence $k = 1 = j$.

Case 2: $j = 1$, $i = 3$. Then $\{x_4\} = F_3 \setminus F_1$, hence $k = 1$.

Case 3: $j = 2$, $i = 3$. Then $\{x_3\} = F_3 \setminus F_2$, hence $k = 2$.

In fact, any ordering of the facets of Δ will be a shelling. The cases above have the following geometric realization shown in Figure 2.11 (cases from left to right); the lighter color vertices are the ones being removed in $F_i \setminus F_j$ for each cases.

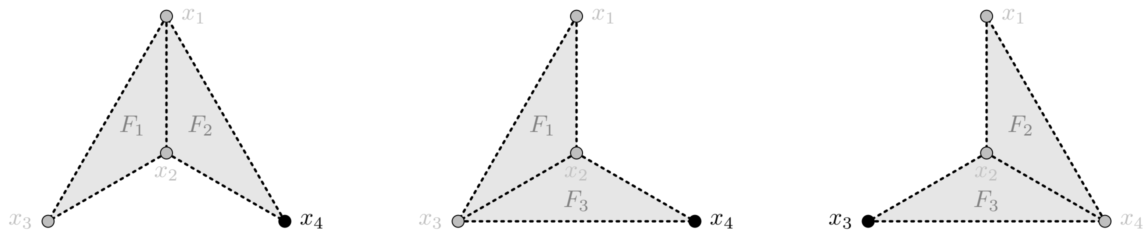


Figure 2.11: Shelling of Δ using condition (b)

Example 2.9.9. Let Δ be a simplicial complex on $V := \{x_1, \ldots, x_5\}$ generated by $F_1 := \{x_1, x_2, x_3\}$ and $F_2 := \{x_1, x_4, x_5\}$; see Figure 2.12. As $F_1 \setminus F_2 = \{x_2, x_3\}$

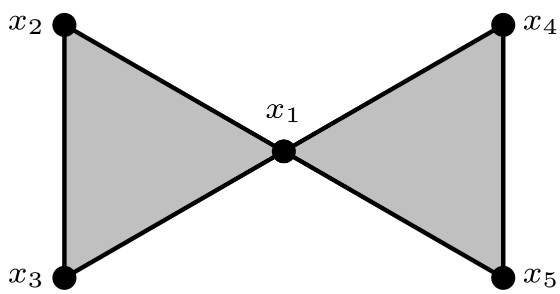


Figure 2.12: Simplicial complex $\Delta = \langle \{x_1, x_2, x_3\}, \{x_1, x_4, x_5\} \rangle$

and $F_2 \setminus F_1 = \{x_4, x_5\}$, no total order on the facets will satisfy condition (b) in Definition 2.9.7. Thus Δ is not shellable.

The following fact explains our interest in shellability. It gives a combinatorial criterion for the Cohen-Macaulay property that is quite useful in practice as we see in Theorem 3.1.17 below.

Fact 2.9.10 ([2, Theorem 5.1.13])**.** The Stanley-Reisner ring of a shellable simplicial complex is Cohen-Macaulay over every field.

Next, we introduce an operation that can be performed on simplicial complexes to obtain another simplicial complex. We will use this operation to prove shellability of a simplicial complex in Theorem 3.1.17 below.

Definition 2.9.11. Let $V' := \{v'_1, \ldots, v'_n\}$ with $V \cap V' = \emptyset$. Let Δ and Δ' be simplicial complexes on V and V', respectively. The *join* of Δ and Δ' is a simplicial complex on $V \cup V'$

$$\Delta * \Delta' := \{U \cup U' : U \in \Delta,\ U' \in \Delta'\}.$$

The following fact shows how the join operator behaves with respect to shellability.

Fact 2.9.12 ([1, Remark 10.22])**.** Let Δ and Δ' be simplicial complexes as in the previous definition. Then $\Delta * \Delta'$ is shellable if and only if Δ and Δ' are shellable.

In particular, Facts 2.9.10 and 2.9.12 are used in [7] to prove that the open neighborhood ideal of any unmixed tree is Cohen-Macaulay by realizing that the open neighborhood ideal of an unmixed tree is a Stanley-Reisner ideal of a shellable simplicial complex called the "ruled complex."

Definition 2.9.13 ([7, page 65])**.** Let $G = (V, E)$ be a graph. Let $\mathcal{D}$ be the set of all minimal TD-sets of G. Set $\mathcal{F}_G = \{V \setminus D : D \in \mathcal{D}\}$. The *ruled complex* of G is the simplicial complex on V generated by the sets in $\mathcal{F}_G$, denoted $\Delta_G := \langle \mathcal{F}_G \rangle$.

Example 2.9.14. Let T be the graph from Example 2.8.2. The set of minimal TD-sets $\mathcal{D}$ of T is

$$\mathcal{D} = \{\{v_1, v_4, v_5, v_7\}, \{v_2, v_4, v_5, v_7\}, \{v_3, v_4, v_5, v_7\}, \{v_4, v_5, v_6, v_7\}\}.$$

Hence the ruled complex of T is

$$\Delta_T = \langle \{v_2, v_3, v_6, v_8\}, \{v_1, v_3, v_6, v_8\}, \{v_1, v_2, v_6, v_8\}, \{v_1, v_2, v_3, v_8\} \rangle .$$

Observe that very facet of Δ_T contains v_8.

Using Fact 2.9.6, we show next that the Stanley-Reisner ideal of the ruled complex of a graph G is the open neighborhood ideal of G.

Theorem 2.9.15. *Let $G = (V, E)$ be a graph and let Δ_G be its ruled complex. Then $J_{\Delta_G} = N(G)$.*

Proof. Let $\mathcal{F}$ be the set of facets of Δ_G and let $\mathcal{D}$ be the set of minimal TD-sets of G. Then we have

$$\begin{aligned} J_{\Delta_G} &= \bigcap_{F \in \mathcal{F}} P_{V \setminus F} && \text{(Fact 2.9.6)} \\ &= \bigcap_{D \in \mathcal{D}} P_D && \text{(Definition 2.9.13)} \\ &= N(G) && \text{(Fact 2.8.16)} \end{aligned}$$

as desired. □

We end the section with a fact stating that the open neighborhood ideal of an unmixed tree is Cohen-Macaulay. It is proved using Fact 2.9.10 and Theorem 2.9.15 by showing that the ruled complex Δ_T is shellable.

Fact 2.9.16 ([7, Theorem 4.3.14])**.** Let $T = (V, E)$ be an unmixed tree and let $R = k[V]$. Then $R/N(T)$ is Cohen-Macaulay over any field.

Chapter 3

Cohen-Macaulay Type of Open Neighborhood Ideals

In this chapter, we compute the type of open neighborhood ideals of unmixed trees, accomplished in Theorem 3.3.9.

Assumptions. Throughout this chapter, k is a field.

3.1 Various Total Dominating Sets

In this section, we generalize the notion of total dominating sets which will be applied to the interior graphs of unmixed trees.

Definition 3.1.1. Let $G = (V, E)$ be a graph and let $S \subseteq V$. A set $D \subseteq V$ is an *S-totally dominating set* of G (*S-TD-set* of G) if $N_G(D) \supseteq S$, and D is a *minimal S-TD-set of* G if there is no proper subset D' of D such that $N_G(D') = S$, i.e., if D does not properly contain another S-TD-set of G.

We define a special type of TD-set for unmixed Δ-trees using the above definition.

Definition 3.1.2. Let $T = (V, E)$ be an unmixed Δ-tree. Set $V_{odd} := V_1 \cup V_3$ and $V_{even} := V_0 \cup V_2$. A set $D \subseteq V$ is a *minimal odd TD-set of* T if D is a minimal V_{odd}-TD-set of T.

Remark 3.1.3. Let T be a tree and let T_B and T_R be the blue and red interior graphs of T, respectively. Since all leaves and isolated vertices of T_B are blue color, we have $\chi(V_0(T_B) \cup V_\infty(T_B)) = \{B\}$. Hence by Fact 2.8.12 with Definition 2.8.5, we get

$$V_{even}(T_B) \subseteq \{v \in V(T_B) : \chi(v) = B\}.$$

We claim that equality holds. Indeed, let $v \in V(T_B)$ such that $\chi(v) = B$. By way of contradiction, assume that height(v) is odd. Then by Fact 2.8.12, leaves of T_B must be colored red, a contradiction. Hence we get the equality

$$V_{even}(T_B) = \{v \in V(T_B) : \chi(v) = B\}.$$

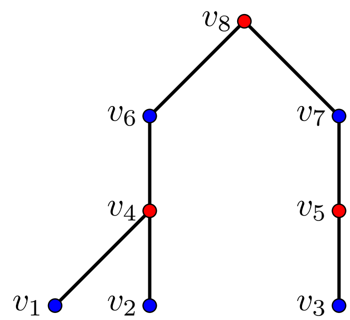
Similarly, we have

$$V_{even}(T_R) = \{v \in V(T_R) : \chi(v) = R\}$$

i.e., the even-height vertices of T_B are the blue vertices of T_B, and the even-height vertices of T_R are the red vertices of T_R.

Example 3.1.4. Consider T from Example 2.8.7. Then we have $V_{odd} = \{v_4, v_5, v_8\}$, the red-colored vertices. It is straightforward to show that the minimal odd TD-sets of T are exactly: $\{v_6, v_7\}$, $\{v_6, v_3\}$, $\{v_1, v_7\}$, and $\{v_2, v_7\}$. On the other hand, the set $\{v_1, v_3\}$ is not an odd TD-set of T since $V_{odd} \not\subseteq N_T(\{v_1, v_3\})$.

Lemma 3.1.5. *Let $T = (V, E)$ be an unmixed Δ-tree. For any minimal TD-set D of T, we have $D \cap V_3 = \emptyset$.*



Proof. Since T is unmixed, we have $\text{height}(T) \in \{0, 1, 3\}$. If $\text{height}(T) < 3$, then the claim holds as $V_3 = \emptyset$. Now suppose that $\text{height}(T) = 3$. Assume by way of contradiction that there exists $v \in D \cap V_3$. Then we have $N(v) \subseteq V_2 \subseteq N(V_1)$. Since V_1 is the set of support vertices of G, we have $V_1 \subseteq D$; then $v \in D \cap V_3$ implies $V_1 \subsetneq D$. Thus, we get $N(D) = N(D \setminus \{v\})$, contradicting the minimality of D. $\square$

Corollary 3.1.6. *Let $T = (V, E)$ be an unmixed Δ-tree. Then every minimal TD-set of T is a union of V_1 and a minimal odd TD-set of T.*

Remark. Corollary 3.1.6 is a direct application of Lemma 3.1.5 and its proof; consider the minimal TD-sets of T' given in Example 2.8.15 for an example.

Definition 3.1.7. Let $T = (V, E)$ be an unmixed Δ-tree. The *odd open neighborhood ideal* of T in $k[V]$ is the ideal

$$N_{odd}(T) := \left\langle \left\{ X_{N(v)} : v \in V_{odd} \right\} \right\rangle .$$

If $V = \{v\}$ (in case $\text{height}(T) = 0$), then $V_{odd} = \emptyset$ so $N_{odd}(T) = 0$.

Example 3.1.8. Let T be the tree from Example 3.1.4. Then $V_{odd} = \{v_4, v_5, v_8\}$. Hence the odd open neighborhood ideal of T in $k[V]$ is given by

$$N_{odd}(T) = \left\langle X_{N(v_4)}, X_{N(v_5)}, X_{N(v_8)} \right\rangle = \langle v_1 v_2 v_6, v_3 v_7, v_6 v_7 \rangle \ .$$

For later use, we give an irredundant m-irreducible decomposition theorem for the "S-open neighborhood ideal," defined next. The proof is almost identical to the one for the "edge ideals" (see [10, Section 4.3]).

Definition 3.1.9. Let $G = (V, E)$ be a graph and let $S \subseteq V$. The S-open neighborhood ideal of G is the ideal in $k[V]$

$$N_S(G) := \left\langle \left\{ X_{N(v)} : v \in S \right\} \right\rangle .$$

Example 3.1.10. Let $T = (V, E)$ be the Δ-tree (forest) T_B from Example 2.8.26 (see Figure 3.1):

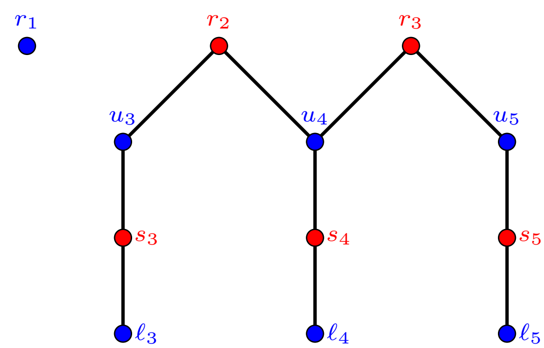


Figure 3.1: T_B from Example 2.8.26

Set $S = V_3 = \{r_2, r_3\}$. (Note that $r_1 \notin V_3$ since height$(r_1) = 0$). Then the S-open neighborhood ideal of T is given by

$$N_S(T) = N_{V_3}(T) = \left\langle \left\{ X_{N(v)} : v \in V_3 \right\} \right\rangle = \langle u_3u_4, u_4u_5 \rangle .$$

Fact 3.1.11 and Lemma 3.1.12 are used in the proof of our decomposition result Theorem 3.1.13.

Fact 3.1.11. Let $G = (V, E)$ be a graph and let $S \subseteq V$. Then

(1) For any $V', V'' \subseteq V$, if V' is an S-TD-set and $V' \subseteq V''$, then V'' is an S-TD-set.

(2) Every S-TD-set contains a minimal S-TD-set.

Proof. This follows readily from the definition of S-TD-set and the finiteness of V. □

Lemma 3.1.12. *Let $G = (V, E)$ be a graph with $V = \{v_1, \ldots, v_d\}$, let $S \subseteq V$, and let $R = k[V]$. Let $V' \subseteq V$. Then V' is an S-TD-set if and only if $N_S(G) \subseteq P_{V'}$.*

Proof. Write $S = \{s_1, \ldots, s_n\}$ with $n \in \mathbb{N}$. For the forward implication, assume that V' is an S-TD-set. Consider $X_{N(s_i)}$ for some $i \in [n]$. Since $s_i \in S$ and V' is an S-TD-set, there exists some vertex $v \in V'$ such that $v \in N(s_i)$. Hence we get $v|X_{N(s_i)}$, so $X_{N(s_i)} \in (v)R \subseteq \langle \{u : u \in V'\} \rangle = P_{V'}$. Thus $N_S(G) \subseteq P_{V'}$.

For the converse implication, assume that $N_S(G) \subseteq P_{V'}$. Let $u \in S$. Then $X_{N(u)} \in N_S(G) \subseteq P_{V'}$. Hence there exists some $v \in V'$ such that $v|X_{N(u)}$. By definition of $X_{N(u)}$, we must have $v \in N(u)$. Thus $u \in N(v) \subseteq N(V')$. Since u is an arbitrary vertex in S, we get $S \subseteq N(V')$. □

Theorem 3.1.13. *Let $G = (V, E)$ be a graph, let $S \subseteq V$, and $R = k[V]$. The S-open neighborhood ideal has the following m-irreducible decomposition*

$$N_S(G) = \bigcap_D P_D = \bigcap_{D \ min} P_D$$

where the first intersection is taken over all S-TD-sets of G, and the second intersection is taken over all minimal S-TD-sets of G. Moreover, the second decomposition is irredundant.

Proof. For any $A, B \subseteq V$, we have $P_A \subseteq P_B$ if and only if $A \subseteq B$. Hence the second intersection is irredundant. Now let $V' \subseteq V$ be an S-TD-set which is not minimal. Then V' contains a minimal S-TD-set by Fact 3.1.11. So, we have $\bigcap_D P_D =$

$\bigcap_{D \neq V'} P_D$. Since V is finite, by repeating the same argument finitely many times, we can conclude that $\bigcap_D P_D = \bigcap_{D\ min} P_D$.

By Lemma 3.1.12, we get $N_S(G) \subseteq \bigcap_D P_D$. For the containment $N_S(G) \supseteq \bigcap_D P_D$, since $N_S(G)$ is a square-free monomial, there are subsets $D_1, \ldots, D_k \subseteq V$ such that $N_S(G) = \bigcap_{i=1}^k P_{D_i}$. For any index $j \in [k]$, we have $N_S(G) \subseteq P_{D_j}$, which implies that D_j is an S-TD-set by Lemma 3.1.12. Thus we get $N_S(G) = \bigcap_{i=1}^k P_{D_i} \supseteq \bigcap_D P_D$. □

Example 3.1.14. Let $T = (V, E)$ be the tree from Example 3.1.10 and set $S = V_3$. Then we have

$$N_S(T) = \langle u_3u_4, u_4u_5 \rangle = \langle u_3, u_5 \rangle \cap \langle u_4 \rangle .$$

The sets $\{u_3, u_5\}$ and $\{u_4\}$ are the minimal S-TD-sets of T.

The following result is a corollary of Theorem 3.1.13 and Corollary 3.1.6 with G being a Δ-tree and $S = V_{odd}$.

Corollary 3.1.15. *Let $T = (V, E)$ be a Δ-tree and $R = k[V]$. The odd open neighborhood ideal of T has the following irredundant m-irreducible decomposition*

$$N_{odd}(T) = \bigcap_{\substack{D' \subseteq V \\ minimal}} P_{D'} = \bigcap_{\substack{D \subseteq V \\ minimal}} P_{D \setminus V_1}$$

where the first intersection is taken over the set of all minimal odd TD-sets of T and the second intersection is taken over the set of all minimal TD-sets of T.

Example 3.1.16. In Example 3.1.8, the odd open neighborhood ideal of T is given by $N_{odd}(T) = \langle v_1v_2v_6, v_3v_7, v_6v_7 \rangle$. By applying Fact 2.6.13, we obtain

$$N_{odd}(T) = \langle v_1, v_7 \rangle \cap \langle v_2, v_7 \rangle \cap \langle v_3, v_6 \rangle \cap \langle v_6, v_7 \rangle .$$

The ideals in the decomposition are exactly the ones generated by minimal odd TD-sets of T found in Example 3.1.4.

Theorem 3.1.17. *Let $T = (V, E)$ be an unmixed Δ-tree (forest). Then the odd open-neighborhood ideal $N_{odd}(T)$ of T is Cohen-Macaulay over any field k.*

Proof. Set

$$\mathcal{D}_{odd} := \{D \subseteq V : D \text{ is a minimal odd TD-set of } T\}$$

and let

$$\mathcal{F}_T := \{V_{even} \setminus D : D \in \mathcal{D}_{odd}\} .$$

Let Δ_T^{odd} be the simplicial complex generated by $\mathcal{F}_T$. Then by Fact 2.9.12 and Corollary 3.1.15, $N_{odd}(T)$ is the Stanley-Reisner ideal of Δ_T^{odd}. Thus, it suffices by Fact 2.9.10 to show that Δ_T^{odd} is shellable. Consider the simplex 2^{V_3}, which is a shellable simplicial complex because it has a single facet V_3. By Corollary 3.1.6, we get $\Delta_T = \Delta_T^{odd} * 2^{V_3}$ where Δ_T is the ruled complex of T from Definition 2.9.13 (the simplicial complex whose Stanley-Reisner ideal is $N(T)$). Fact 2.9.16 says Δ_T is shellable by the unmixedness of T. Hence the complex Δ_T^{odd} is shellable by Fact 2.9.12. $\square$

Example 3.1.18. Here is an example that demonstrates the proof of Theorem 3.1.17. Let T be the tree from Example 3.1.4. Then we have

$$\mathcal{D}_{odd} = \{\{v_1, v_7\}, \{v_2, v_7\}, \{v_3, v_7\}, \{v_6, v_7\}\}.$$

Hence the simplicial complex Δ_T^{odd} is given by

$$\Delta_T^{odd} = \langle \{v_2, v_3, v_6\}, \{v_1, v_3, v_6\}, \{v_1, v_2, v_6\}, \{v_1, v_2, v_3\} \rangle .$$

By Example 2.9.14, the ruled complex of T is given by

$$\Delta_T = \langle \{v_2, v_3, v_6, v_8\}, \{v_1, v_3, v_6, v_8\}, \{v_1, v_2, v_6, v_8\}, \{v_1, v_2, v_3, v_8\} \rangle .$$

The facets of Δ_T are the facets of Δ_T^{odd} unioned with $\{v_8\}$. Thus we have

$$\Delta_T = \Delta_T^{odd} * \langle \{v_8\} \rangle = \Delta_T^{odd} * 2^{\{v_8\}} = \Delta_T^{odd} * 2^{V_3}$$

as $V_3 = \{v_8\}$ in T.

The following corollary will be used in Lemmas 3.2.3 and 3.2.6.

Corollary 3.1.19. *Let $T = (V, E)$ be an unmixed Δ-tree. Then $k[V_{even}]/N_{odd}(T)$ is also Cohen-Macaulay.*

Proof. We know that $k[V]/N_{odd}(T)$ is Cohen-Macaulay by Theorem 3.1.17. Because the generators of $N_{odd}(T)$ only use V_{even}, the set of variables V_{odd} forms a $k[V]/N_{odd}(T)$-regular sequence. Thus $k[V]/(N_{odd}(T) + \langle V_{odd} \rangle) \cong k[V_{even}]/N_{odd}$ is Cohen-Macaulay by Fact 2.7.5. □

3.2 Cohen-Macaulay Type of Δ-trees

We will use the Cohen-Macaulay type results for unmixed Δ-trees to compute the type of general unmixed trees.

Assumptions. In this section, we assume that $T = (V, E)$ is an unmixed Δ-tree, and set $R = k[V_{even}]$ and $(V_{even})R =: \mathfrak{X}$, unless otherwise stated. If T is a graph with single vertex v, then set $R := k[v]$ and $\mathfrak{X} := (v)R$.

We begin with a vertex labeling that will be used throughout the proofs in this chapter for simplicity.

Notation 3.2.1. Assume T has height 3. For $\ell = 0, 1, 2, 3$, set $n_\ell := |V_\ell|$. Theorem 2.8.21 and Remark 2.8.22 allow us to denote the vertices of T as follows:

(1) write $V_1 := \{s_1, \ \ldots \ , s_{n_1}\}$ and $V_3 := \{r_1, \ \ldots \ , r_{n_3}\}$;

(2) for $i \in [k]$ write u_i for the unique height 2 vertex adjacent to s_i; and

(3) write $\ell_{i,1}, \ \ldots \ , \ell_{i,m_i}$ for the leaves adjacent to s_i ($m_i \geq 1$).

Since T is unmixed of height 3, we have $n_1 = n_2$ by Remark 2.8.22.

Example 3.2.2. Using Notation 3.2.1, we can label the vertices of an unmixed Δ-tree T of height 3 as in Figure 3.2.

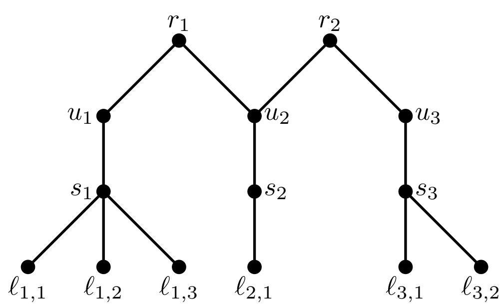


Figure 3.2: Example of a vertex labeling on a Δ-tree using Notation 3.2.1.

Here, $n_1 = |V_1| = 3$, $n_3 = |V_3| = 2$, $m_1 = 3$, $m_2 = 1$, and $m_3 = 2$. The labeling rule (2) in Notation 3.2.1 is well-defined by Theorem 2.8.21. Also, we have $|V_1| = |V_2| = n_2$ (for height-3 unmixed Δ-trees), $|V_3| = n_3$, and for $i \in [n_1]$, there are m_i leaves adjacent to s_i. Thus, we have

$$|V_{even}| = n_2 + \sum_{i=1}^{n_2} m_i = n_2 + n_0 \qquad (3.1)$$

e.g., for the tree in Figure 3.2, we have

$$9 = |V_{even}| = 3 + (3 + 1 + 2) = 3 + 6.$$

To compute type($R/N_{odd}(T)$), we first identify a maximal $(R/N_{odd}(T))$-regular sequence. First, we consider unmixed Δ-trees of height 0 and 1.

Lemma 3.2.3. *Suppose that height(T) ≤ 1. Let $Z \subseteq R$ be defined as followings:*

(1) If height(T) $= 0$, set $V = \{v\}$ and $Z := \{v\}$;

(2) If height(T) $= 1$, set $V_1 = \{s\}$, $V_0 = \{\ell_1, \ldots, \ell_{n_0}\}$ with $n_0 \geq 2$, and $Z := \{\ell_1 - \ell_j : 2 \leq j \leq n_0\}$.

Then Z is a maximal $R/N_{odd}(T)$-regular sequence in $\mathfrak{X}$.

Proof. (1) If height(T) $= 0$, then T is a graph with a single vertex, and we have $R = k[v]$ and $N_{odd}(T) = 0$. Thus $R/N_{odd}(T) \cong R = k[v]$ and $\{v\} = Z$ is a maximum regular sequence for $k[v]$ by Example 2.5.9.

(2) Now suppose that height(T) $= 1$. We use Fact 2.7.5 to show that Z is a maximal $R/N_{odd}(T)$-regular sequence. We have $R = k[V_{even}] = k[\ell_1, \ldots, \ell_{n_0}]$ and $N_{odd}(T) = \left\langle X_{N(s)} \right\rangle = \langle \ell_1 \cdots \ell_{n_0} \rangle$. Since the m-irreducible decomposition of $N_{odd}(T)$ is

$$N_{odd}(T) = \langle \ell_1 \cdots \ell_{n_0} \rangle = \bigcap_{i=1}^{n_0} \langle \ell_i \rangle ,$$

we have $\dim(R/N_{odd}(T)) = n_0 - 1$ by Fact 2.6.15. Then we have

$$\frac{R}{N_{odd}(T) + \langle Z \rangle} = \frac{k[\ell_1, \ldots, \ell_{n_0}]}{\langle \ell_1 \cdots \ell_{n_0} \rangle + \langle \{\ell_1 - \ell_j : 2 \leq j \leq n_0\} \rangle} \cong \frac{k[\ell_1]}{\langle \ell_1^{n_0} \rangle}.$$

Since the ideal $\langle \ell_1^{n_0} \rangle$ is an m-irreducible decomposition, we get that $\dim(k[\ell_1]/\langle \ell_1^{n_0} \rangle) = 1 - 1 = 0$ by Fact 2.6.15. Since $R/N_{odd}(T)$ is Cohen-Macaulay by Corollary 3.1.19 and $|Z| = n_0 - 1$, the elements in Z form a maximal $(R/N_{odd}(T))$-regular sequence by Fact 2.7.5. □

The following is a quick corollary of Lemma 3.2.3.

Corollary 3.2.4. *Suppose that height$(T) \leq 1$. Then*

$$depth(R/N_{odd}(T)) = \begin{cases} 1 & \text{if } height(T) = 0 \\ |V_0| - 1 & \text{if } height(T) = 1 \end{cases}.$$

Next, we consider unmixed Δ-trees of height 3; recall that no height-2 unmixed Δ-tree exists by Example 2.8.23. We begin with a short lemma about the depth of $R/N_{odd}(T)$ when height$(T) = 3$.

Lemma 3.2.5. *Let height$(T) = 3$. Using Notation 3.2.1, we have*

$$depth(R/N_{odd}(T)) = n_0.$$

Proof. Recall that $R = k[V_{even}]$. Since T is unmixed, Corollary 3.1.6 implies that every minimal odd TD-set of T has the same size; moreover, they all have size n_2 since the set V_2 is a minimal odd TD-set of T. Thus we get

$$\begin{aligned} \text{depth}(R/N_{odd}(T)) &= \dim(R/N_{odd}(T)) && \text{(Corollary 3.1.19)} \\ &= |V_{even}| - n_2 && \text{(Fact 2.6.15)} \\ &= \sum_{i=1}^{n_2} m_i && \text{(Example 3.2.2)} \\ &= \sum_{i=1}^{n_1} m_i && (n_1 = n_2) \\ &= n_0. && \square \end{aligned}$$

Now we exhibit a maximal $R/N_{odd}(T)$-regular sequence when height$(T) = 3$.

Lemma 3.2.6. *Let height*$(T) = 3$. *Set*

$$Z := \{u_i - \ell_{i,j} : 1 \le i \le n_1, 1 \le j \le m_i\} \subseteq R$$

Then Z is a maximal $R/N_{odd}(T)$-regular sequence in $\mathfrak{X}$.

Proof. Set $R' := k[u_1, \dots, u_{n_2}]$. Then we have

$$\frac{R}{N_{odd}(T) + \langle Z \rangle} \cong \frac{R'}{\left(u_1^{|N(s_1)|}, \dots, u_{n_2}^{|N(s_{n_1})|}, X_{N(r_1)}, \dots, X_{N(r_{n_3})}\right) R'} \qquad (*)$$

because every leaf $\ell_{i,j}$ is identified with its closest height 2 vertex u_i and $n_1 = |V_1| = |V_2| = n_2$ by Remark 2.8.22 since T is an unmixed Δ-tree of height 3. Set

$$\mathfrak{I} := \left(u_1^{|N(s_1)|}, \dots, u_{n_1}^{|N(s_{n_1})|}, X_{N(r_1)}, \dots, X_{N(r_{n_3})}\right) R'.$$

Since $\mathfrak{I}$ contains positive powers of all the u_i's, and the $X_{N(r_i)}$ are non-unit monomials, we get m-rad$(\mathfrak{I}) = (u_1, \dots, u_{n_1})R'$. This implies that $\dim(R'/\mathfrak{I}) = 0$ by Fact 2.6.15. Since $R/N_{odd}(T)$ is Cohen-Macaulay of depth n_0 by Corollary 3.1.19 and Lemma 3.2.5, the condition

$$|Z| = \sum_{i=1}^{n_2} m_i = n_0,$$

implies that Z is a maximal regular sequence by Fact 2.7.5. □

Example 3.2.7. Let T be the tree from Example 3.2.2 and let $R = k[V_{even}]$; see Figure 3.3, some vertices are colored for guidance. Then the maximal $R/N_{odd}(T)$-regular sequence from Lemma 3.2.6 is

$$Z = \{u_1 - \ell_{1,1}, u_1 - \ell_{1,2}, u_1 - \ell_{1,3}, u_2 - \ell_{2,1}, u_3 - \ell_{3,1}, u_3 - \ell_{3,2}\}.$$

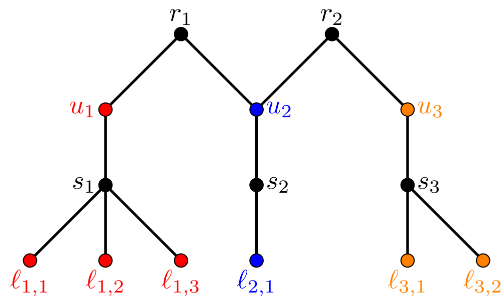


Figure 3.3: Tree T from Example 3.2.2

We have

$$N_{odd}(T) = \langle u_1\ell_{1,1}\ell_{1,2}\ell_{1,3}, u_2\ell_{2,1}, u_3\ell_{3,1}\ell_{3,2}, u_1u_2, u_2u_3\rangle$$

hence

$$\frac{R}{N_{odd}(T) + \langle Z\rangle} = \frac{k[u_1, u_2, u_3, \ell_{1,1}, \ell_{1,2}, \ell_{1,3}, \ell_{2,1}, \ell_{3,1}, \ell_{3,2}]}{\langle u_1\ell_{1,1}\ell_{1,2}\ell_{1,3}, u_2\ell_{2,1}, u_3\ell_{3,1}\ell_{3,2}, u_1u_2, u_2u_3\rangle + \langle Z\rangle}$$

$$= \frac{k[u_1, u_2, u_3]}{\langle {u_1}^4, {u_2}^2, {u_3}^3, u_1u_2, u_2u_3\rangle}.$$

The ideal $\langle {u_1}^4, {u_2}^2, {u_3}^3, u_1u_2, u_2u_3\rangle$ is the ideal $\mathfrak{J}$ from the proof of Lemma 3.2.6.

Now we compute the type of $R/N_{odd}(T)$ when T is an unmixed Δ-tree. We first take care of the cases when height$(T) < 3$. Note that this result says that $R/N_{odd}(T)$ is "Gorenstein" in this case.

Theorem 3.2.8. *If height$(T) < 3$, then type$(R/N_{odd}(T)) = 1$.*

Proof. First, let height$(T) = 0$; hence we write $V = \{v\}$. Then $R = k[v]$ and $N_{odd}(T) = 0$. Hence we have

$$\begin{aligned} \text{type}(R/N_{odd}(T)) &= \text{type}(R) && (N_{odd}(T) = 0) \\ &= \text{type}(R/\langle v\rangle) = 1. && \text{(Facts 2.7.7 and 2.7.10)} \end{aligned}$$

Now suppose that height$(T) = 1$. Let Z be the maximal $R/N_{odd}(T)$-regular sequence from Lemma 3.2.3. Then we have

$$\text{type}(R/N_{odd}(T)) = \text{type}\left(\frac{R}{N_{odd}(T) + \langle Z \rangle}\right) \qquad \text{(Fact 2.7.7)}$$

$$= \text{type}(k[\ell_{1,1}]/\left\langle \ell_{1,1}^{m_1} \right\rangle) \qquad \text{((1) in Lemma 3.2.3)}$$

$$= 1 \qquad \text{(Fact 2.7.10)}$$

as desired. □

Now we consider T with height$(T) = 3$. The ideal $\mathfrak{J}$ from Lemma 3.2.6 has a parametric decomposition in $k[u_1, \dots, u_{n_2}]$ since m-rad$(\mathfrak{J}) = \langle u_1, \dots, u_{n_2} \rangle$. Lemma 3.2.9 gives us the decomposition of $\mathfrak{J}$ explicitly.

Lemma 3.2.9. *Assume that height$(T) = 3$. Let $\mathfrak{J} \leq R' = k[u_1, \dots, u_{n_2}]$ be the ideal from the proof of Lemma 3.2.6. Set*

$$U := \left(u_1^{|N(s_1)|}, \ \dots \ , u_{n_2}^{|N(s_{n_1})|}\right) R'.$$

The irredundant parametric decomposition of $\mathfrak{J}$ in R' is

$$\mathfrak{J} = \bigcap_{D \ min} (P_D + U)$$

where the intersection is taken over all minimal V_3-TD-sets of T.

Proof. Notice that the ideal $\left(X_{N(r_1)}, \dots, X_{N(r_{n_3})}\right) R'$ is the V_3-open neighborhood ideal of T since $V_3 = \{r_1, \ldots, r_{n_3}\}$. Thus by Theorem 3.1.13, we get

$$\left(X_{N(r_1)}, \ldots, X_{N(r_{n_3})}\right) R' = \bigcap_{D\ min} P_D$$

where the intersection is taken over all minimal V_3-TD-set of T. Thus we get

$$\mathfrak{J} = \left(X_{N(r_1)}, \ldots, X_{N(r_{n_3})}\right) R' + U = \left(\bigcap_{D\ min} P_D\right) + U = \bigcap_{D\ min} (P_D + U)$$

where the last equality comes from Lemma 2.6.29. □

Example 3.2.10. Let $T = (V, E)$ be the tree from Example 3.2.2 and let $\mathfrak{J}$ be the ideal from Example 3.2.7. Setting

$$U = \left\langle u_1^{|N(s_1)|}, u_2^{|N(s_2)|}, u_3^{|N(s_3)|} \right\rangle = \left\langle u_1^4, u_2^2, u_3^3 \right\rangle,$$

we get

$$\begin{aligned} \mathfrak{J} &= \left\langle u_1^4, u_2^2, u_3^3, u_1u_2, u_2u_3 \right\rangle \\ &= \left\langle u_1^4, u_2^2, u_3^3, u_1, u_3 \right\rangle \cap \left\langle u_1^4, u_2^2, u_3^3, u_2 \right\rangle \\ &= \left(\langle u_1, u_3 \rangle + U\right) \cap \left(\langle u_2 \rangle + U\right). \end{aligned}$$

Sets $\{u_1, u_3\}$ and $\{u_2\}$ are minimal V_3-TD-sets of T.

Now we can describe the type$(R/N_{odd}(T))$ for an unmixed Δ-tree of any height using minimal V_3-TD-sets.

Theorem 3.2.11. *Let $R = k[V_{even}]$. Then*

$$type(R/N_{odd}(T)) = \text{“}\textit{number of minimal } V_3\textit{-TD-sets of } T.\text{”}$$

Proof. If height$(T) < 3$, then the empty set is the unique minimal V_3-TD-set, so

$$\begin{aligned} \operatorname{type}(R/N_{odd}(T)) &= 1 && \text{(Theorem 3.2.8)} \\ &= |\{\emptyset\}| \\ &= \text{“number of minimal } V_3\text{-TD-sets of } T.\text{”} \end{aligned}$$

Now suppose that height$(T) = 3$. Let Z, R', and $\mathfrak{I}$ as in Lemma 3.2.6. Then we have

$$\begin{aligned} \operatorname{type}(R/N_{odd}(T)) &= \operatorname{type}\left(\frac{R}{N_{odd}(T) + \langle Z \rangle}\right) && \text{(Fact 2.7.7)} \\ &= \operatorname{type}(R'/\mathfrak{I}) && \text{((}*\text{) in Lemma 3.2.6)} \\ &= \text{“number of ideals in the parametric decomp. of } \mathfrak{I}\text{”} && \text{(Fact 2.7.10)} \\ &= \text{“number of minimal } V_3\text{-TD-sets of } T\text{”} && \text{(Lemma 3.2.9)} \end{aligned}$$

as desired. □

Example 3.2.12. Let $T = (V, E)$ be the tree from Example 3.2.2. Then by Theorem 3.2.11 with Example 3.2.10, we get

$$\operatorname{type}(k[V_{even}]/N_{odd}(T)) = \text{“number of minimal } V_3\text{-TD-set of } T\text{”} = 2.$$

3.3 Cohen-Macaulay Type of Unmixed Trees

Assumptions. Let $T = (V, E)$ be an unmixed tree and let T_R and T_B be the two interior trees (forests) of T derived from a 2-coloring χ; see Definition 2.8.5. Set $N_B(v) := N_{T_B}(v)$ and $N_R(v) := N_{T_R}(v)$ for all $v \in V$.

Theorem 3.3.1 gives us a connection between open neighborhood ideal of T and the odd open neighborhood ideals of T's interior graphs.

Theorem 3.3.1. *We have*

$$\begin{aligned} N(T) &= N_{odd}(T_B)R + N_{odd}(T_R)R + P_{V_1(T)} \\ &= N_{odd}(T_B)R + N_{odd}(T_R)R + (V_1(T))R. \end{aligned}$$

Proof. ($\subseteq$) It suffices to show that $X_{N_T(v)} \in N_{odd}(T_B)R + N_{odd}(T_R)R + P_{V_1(T)}$ for all $v \in V$. Let $v \in V$. By symmetry, assume that $\chi(v) = R$.

Case 1: $v \notin V(T_B)$. Then by Definition 2.8.25, there exists some blue support vertex $s \in V(T)$ that is adjacent to v. Suppose that v is a leaf in T. Then $N_T(v) = \{s\}$. Since $s \in P_{V_1(T)}$, we have $X_{N_T(v)} = s \in P_{V_1(T)}$. Now suppose that v is not a leaf. Since s is a support vertex, there exists some leaf $\ell \in V(T)$ that is adjacent to s. Then $X_{N_T(\ell)} = s$ divides $X_{N_T(v)}$ since $s \in N_T(v)$. Hence $X_{N_T(v)}$ is a redundant generator of $N(T)$.

Case 2: $v \in V(T_B)$. Then $N_{T_B}(v) \subseteq N_T(v)$. Thus $X_{N_B(v)} | X_{N_T(v)}$, so $X_{N_T(v)} \in N(T_B)$.

($\supseteq$) Every support vertex of T is a generator of $N(T)$, being the open neighborhood of each leaf it is adjacent to, so we get $P_{V_1(T)} \subseteq N(T)$. To complete the proof, by symmetry, we show that $N(T) \supseteq N_{odd}(T_B)$. Note that all we need to show is that

$X_{N_B(v)} \in N(T)$ for all $v \in V_{odd}(T_B)$ as this shows that the generators of $N_{odd}(T_B)$ are in $N(T)$. So, let $v \in V_{odd}(T_B)$. We show that $N_B(v) = N_T(v)$ which will imply that $X_{N_B(v)} = X_{N_T(v)} \in N(T)$. To this end, assume by way of contradiction that $N_B(v) \subsetneq N_T(v)$. Since height$(v) \in \{1,3\}$ by the definition of $V_{odd}(T_B)$ along with Facts 2.8.18 and 2.8.29, we get $\chi(v) = R$ as $v \in V(T_B)$. Since $N_B(v) \subsetneq N_T(v)$, there exists a vertex $u \in V$ such that $u \in N_T(v) \setminus V(T_B)$ and $uv \in E(T)$. Since $\chi(v) = R$, we have $\chi(u) = B$. Since $u \notin V(T_B)$, u must be a vertex that is deleted from T while constructing T_B. By definition of T_B, u must be a support vertex since $\chi(u) = B$. But, this implies that $v \notin V(T_B)$ since v must be deleted from T while constructing T_B as $v \in N_T(u)$, a contradiction. □

Example 3.3.2. Let T, T_B, and T_R be the graphs from Example 2.8.26 and set $R = k[V]$. Then we have

$$N_{odd}(T_B) = \langle \ell_3u_3, \ell_4u_4, \ell_5u_5, u_3u_4, u_4u_5\rangle\,,$$
$$N_{odd}(T_R) = \langle \ell_1u_1, \ell_2u_2, u_1u_2\rangle\,,$$
$$(V_1(T))R = \langle s_1, s_2, s_3, s_4, s_5\rangle\,.$$

The variables are colored using the same coloring of T from Example 2.8.26. The open neighborhood ideal of T is given by

$$\begin{aligned} N(T) = \langle & s_1, s_2, s_3, s_4, s_5, \ell_1u_1, \ell_2u_2, \ell_3u_3, \ell_4u_4, \ell_5u_5, s_1r_1, s_2u_3r_1, s_3u_2r_2, s_4r_2r_3, \\ & s_5r_3, u_1u_2, u_3u_4, u_4u_5\rangle \\ = \langle & s_1, s_2, s_3, s_4, s_5, \ell_1u_1, \ell_2u_2, \ell_3u_3, \ell_4u_4, \ell_5u_5, u_1u_2, u_3u_4, u_4u_5\rangle\,. \end{aligned}$$

Notice that the minimal set of generators of $N(T)$ is exactly the union of the generating sets for $N_{odd}(T_B)$, $N_{odd}(T_R)$, and $(V_1(T))R$. Thus we get

$$N(T) = N_{odd}(T_B) + N_{odd}(T_R) + (V_1(T))R.$$

Remark. By the construction of the interior graphs T_B and T_R, we have $N_{odd}(T_B) \subseteq (\chi^{-1}(B) \setminus V_1(T))R$ and $N_{odd}(T_R) \subseteq (\chi^{-1}(R) \setminus V_1(T))R$; that is, the variables of the generators of $N_{odd}(T_B)$ and $N_{odd}(T_R)$ are blue and red, respectively. Hence the ideals $N_{odd}(T_B)$, $N_{odd}(T_R)$, and $(V_1(T))R$ use pairwise disjoint sets of variables for their generators.

The following lemma tells us how the vertex set of T is partitioned by the vertices of its interior graphs which is used in Theorem 3.3.9.

Lemma 3.3.3. *The set $V(T)$ can be partitioned by*

$$V(T) = V_{even}(T_B) \cup V_{even}(T_R) \cup V_1(T).$$

Proof. The inclusion $V(T) \supseteq V_{even}(T_B) \cup V_{even}(T_R) \cup V_1(T)$ is by construction, so we consider the other inclusion. Let $v \in V(T)$. By symmetry, assume that $\chi(v) = B$; i.e., v is colored blue. Suppose that $v \in V(T_B)$. Then by Remark 3.1.3, $v \in V_{even}(T_B)$. So suppose that $v \notin V(T_B)$. Then by the construction of T_B, the blue vertex v is a support vertex in T since we only delete the blue support vertices of T and their red neighbors to obtain T_B. Thus $v \in V_1(T)$. Thus we get $V(T) \subseteq V_{even}(T_B) \cup V_{even}(T_R) \cup V_1(T)$.

Now we show that the sets $V_{even}(T_B)$, $V_{even}(T_R)$, and $V_1(T)$ are pairwise disjoint. By Remark 3.1.3, $V_{even}(T_B)$ is colored blue and $V_{even}(T_R)$ is colored red; hence $V_{even}(T_B) \cap V_{even}(T_R) = \emptyset$. By symmetry, it suffices to show that $V_1(T) \cap V_{even}(T_B) = \emptyset$. Let $v \in V_1(T)$. If $\chi(v) = B$, then $v \notin V(T_B) \subseteq V_{even}(T_B)$ by construction of T_B. If

$\chi(v) = R$, then $v \notin V_{even}(T_B)$ since every vertex in $V_{even}(T_B)$ is colored blue. $\square$

Example 3.3.4. Consider the trees from Example 2.8.27. We have

$$V_{even}(T_B) = \{v_1, v_2, v_4, v_5, v_9, v_{14}, v_{19}, v_{20}, v_{24}\}$$

$$V_{even}(T_R) = \{v_3, v_6, v_7, v_{10}, v_{15}, v_{18}, v_{21}, v_{22}, v_{23}\}$$

$$V_1(T) = \{v_8, v_{11}, v_{12}, v_{13}, v_{16}, v_{17}\}$$

Since $V(T) = \{v_1, \ldots, v_{24}\}$, we can see that $V(T)$ is partitioned by the three sets above.

The following result is used for Corollary 3.3.6 which shows that the odd open neighborhood ideal of an unmixed Δ-forest is also Cohen-Macaulay. The depth computation is used in Theorem 3.3.7.

Lemma 3.3.5. *Let T_1 and T_2 be unmixed Δ-trees with $V(T_1) \cap V(T_2) = \emptyset$. Let $T = (V(T_1) \cup V(T_2), E(T_1) \cup E(T_2))$. Then $k[V_{even}(T)]/N_{odd}(T)$ is Cohen-Macaulay over any field, and*

$$depth\left(\frac{k[V_{even}(T)]}{N_{odd}(T)}\right) = depth\left(\frac{k[V_{even}(T_1)]}{N_{odd}(T_1)}\right) + depth\left(\frac{k[V_{even}(T_2)]}{N_{odd}(T_2)}\right).$$

Proof. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the set of all minimal odd TD-sets of T_1 and T_2, respectively. Let $\mathcal{F}_1 := \{V_{even}(T_1) \setminus D : D \in \mathcal{D}_1\}$ and $\mathcal{F}_2 := \{V_{even}(T_2) \setminus D : D \in \mathcal{D}_2\}$. Let Δ_1 and Δ_2 be simplicial complexes generated by $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively. In the proof of Theorem 3.1.17, we showed that Δ_1 and Δ_2 are shellable. Now let $\mathcal{D}$ be the set of all minimal odd TD-sets of T, let $\mathcal{F} = \{V_{even}(T) \setminus D : D \in \mathcal{D}\}$, and let Δ be the simplicial complex generated by $\mathcal{F}$. By the construction of T, the minimal odd TD-sets of T are the unions of the minimal odd TD-sets of T_1 and T_2. So, we have

$$\mathcal{D} = \{D_1 \cup D_2 : D_1 \in \mathcal{D}_1, D_2 \in \mathcal{D}_2\}$$

hence

$$\mathcal{F} = \{F_1 \cup F_2 : F_1 \in \mathcal{F}_1, F_2 \in \mathcal{F}_2\}\,.$$

This implies that $\Delta = \Delta_1 * \Delta_2$, so Δ is shellable by Fact 2.9.12. Therefore, by Fact 2.9.10, $k[V_{even}(T)]/N_{odd}(T)$ is Cohen-Macaulay.

Next, we show the equality for depth. Let $D \in \mathcal{D}$, and let $D_1 \in \mathcal{D}_1$ and $D_2 \in \mathcal{D}_2$ be so that $D = D_1 \cup D_2$. Since T is unmixed, we get

$$\dim\left(\frac{k[V_{even}(T)]}{N_{odd}(T)}\right) = |V_{even}(T)| - |D|$$

by Corollary 3.1.15 and Fact 2.6.15. Similarly, for $i \in \{1, 2\}$ we have

$$\dim\left(\frac{k[V_{even}(T_i)]}{N_{odd}(T_i)}\right) = |V_{even}(T_i)| - |D_i|.$$

Thus we get

$$\begin{aligned}\dim\left(\frac{k[V_{even}(T)]}{N_{odd}(T)}\right) &= |V_{even}(T)| - |D| \\ &= |V_{even}(T_1) \cup V_{even}(T_2)| - |D_1 \cup D_2| \\ &= |V_{even}(T_1)| + |V_{even}(T_2)| - |D_1| - |D_2| \\ &= \dim\left(\frac{k[V_{even}(T_1)]}{N_{odd}(T_1)}\right) + \dim\left(\frac{k[V_{even}(T_2)]}{N_{odd}(T_2)}\right).\end{aligned}$$

Since the odd open neighborhood ideal for T, T_1, and T_2 are all Cohen-Macaulay, we get

$$\operatorname{depth}\left(\frac{k[V_{even}(T)]}{N_{odd}(T)}\right) = \dim\left(\frac{k[V_{even}(T)]}{N_{odd}(T)}\right)$$

$$= \dim\left(\frac{k[V_{even}(T_1)]}{N_{odd}(T_1)}\right) + \dim\left(\frac{k[V_{even}(T_2)]}{N_{odd}(T_2)}\right)$$

$$= \operatorname{depth}\left(\frac{k[V_{even}(T_1)]}{N_{odd}(T_1)}\right) + \operatorname{depth}\left(\frac{k[V_{even}(T_2)]}{N_{odd}(T_2)}\right). \qquad \square$$

Corollary 3.3.6. *Let T be an unmixed Δ-forest with connected components $T_1,\ldots,T_c$. Then $k[V_{even}(T)]/N_{odd}(T)$ is Cohen-Macaulay and*

$$depth\left(\frac{k[V_{even}(T)]}{N_{odd}(T)}\right) = \sum_{i=1}^{c} depth\left(\frac{k[V_{even}(T_i)]}{N_{odd}(T_i)}\right).$$

Proof. Induct on c, and apply Lemma 3.3.5. $\square$

Next, we state and proof a result on a maximal $k[V_{even}]/N_{odd}(T)$-regular sequence of an unmixed Δ-forest T that is used in Theorem 3.3.9.

Theorem 3.3.7. *Let T be an unmixed Δ-forest with connected components $T_1,\ldots,T_c$. Let $R = k[V_{even}]$ and $R_i = k[V_{even}(T_i)]$ for all $i \in [c]$. For each $i \in [c]$, using Notation 3.2.1 for each T_i, let Z_i be the maximal regular sequence constructed in Lemmas 3.2.3 and 3.2.6. Set $Z = \bigcup_{i=1}^{c} Z_i$. Then*

(1) Z is a maximal $R/N_{odd}(T)$-regular sequence in $\langle V_{even}\rangle$.

(2) type$(R/N_{odd}(T))$ = "number of minimal V_3-TD-sets in T."

Proof. Since T is a forest with connected components $T_1,\ldots,T_c$, we have $V_{odd} = \bigcup_{i=1}^{c} V_{odd}(T_i)$ and $V_{even} = \bigcup_{i=1}^{c} V_{even}(T_i)$. So, by Definition 3.1.7, we get

$$\begin{aligned} N_{odd}(T) &= \left\langle \left\{ X_{N_T(v)} : v \in V_{odd}(T) \right\} \right\rangle \\ &= \left\langle \bigcup_{i=1}^{c} \left\{ X_{N_{T_i}(v)} : v \in V_{odd}(T_i) \right\} \right\rangle \end{aligned}$$

$$= \sum_{i=1}^{c} \left\langle \left\{ X_{N_{T_i}(v)} : v \in V_{odd}(T_i) \right\} \right\rangle$$

$$= \sum_{i=1}^{c} N_{odd}(T_i).$$

Hence we have

$$\frac{R}{N_{odd}(T) + \langle Z \rangle} = \frac{R}{\sum_{i=1}^{c} N_{odd}(T_i) + \sum_{j=1}^{c} \langle Z_j \rangle}$$

$$= \frac{R}{\sum_{i=1}^{c} \left(N_{odd}(T_i) + \langle Z_i \rangle\right)}$$

$$= \bigotimes_{i=1}^{c} \frac{R_i}{N_{odd}(T_i) + \langle Z_i \rangle} \qquad \text{(Example 2.4.4)}$$

where the tensor product is taken over k. Since $\dim\left(R_i/(N_{odd}(T_i) + \langle Z_i \rangle)\right) = 0$ for all i, we get $\dim(R/(N_{odd}(T) + \langle Z \rangle)) = 0$ by Lemma 2.6.30. Note that we have

$$\operatorname{depth}\left(\frac{k[V_{even}(T)]}{N_{odd}(T)}\right) = \sum_{i=1}^{c} \operatorname{depth}\left(\frac{k[V_{even}(T_i)]}{N_{odd}(T_i)}\right) \qquad \text{(Corollary 3.3.6)}$$

$$= \sum_{i=1}^{c} |Z_i| \qquad \text{(definition of } Z_i\text{)}$$

$$= |Z|.$$

So, $k[V_{even}(T)]/N_{odd}(T)$ is Cohen-Macaulay (Corollary 3.3.6) of depth $|Z|$, hence Z is a maximal regular sequence by Fact 2.7.5.

Noting that $R \cong \otimes_{i=1}^{c} R_i$ (tensoring over k), the type of $R/N_{odd}(T)$ is given by

$$\begin{aligned}
\operatorname{type}\left(\frac{R}{N_{odd}(T)}\right) &= \operatorname{type}\left(\bigotimes_{i=1}^{c}\frac{R_i}{N_{odd}(T_i)}\right)\\
&= \dim_k\left(\operatorname{Ext}^0_{\otimes_{i=1}^{c}R_i}\left(k, \bigotimes_{i=1}^{c}\frac{R_i}{N_{odd}(T_i)}\right)\right)\\
&= \dim_k\left(\bigotimes_{i=1}^{c}\operatorname{Ext}^0_{R_i}\left(k, \frac{R_i}{N_{odd}(T_i)}\right)\right) \qquad \text{(Fact 2.4.5)}\\
&= \prod_{i=1}^{c}\dim_k\left(\operatorname{Ext}^0_{R_i}\left(k, \frac{R_i}{N_{odd}(T_i)}\right)\right)\\
&= \prod_{i=1}^{c}\operatorname{type}(R_i/N_{odd}(T_i)).
\end{aligned}$$

By Theorem 3.2.11, $\operatorname{type}(R_i/N_{odd}(T_i))$ is given by the number of minimal $V_3(T_i)$-TD-sets of T_i for each i. Since T is the union of the connected components $T_1, \ldots, T_c$, the number of minimal V_3-TD-set of T is the product of the number of minimal $V_3(T_i)$-TD-sets of T_i for all i. Thus we get

$$\begin{aligned}
\operatorname{type}\left(\frac{R}{N_{odd}(T)}\right) &= \prod_{i=1}^{c}\operatorname{type}(R_i/N_{odd}(T_i))\\
&= \text{“number of minimal } V_3\text{-TD-sets in } T\text{”}. \qquad \square
\end{aligned}$$

Example 3.3.8. We demonstrate the computations in Theorem 3.3.7. Consider T_B from Example 2.8.27 shown below in Figure 3.4. For each connected component,

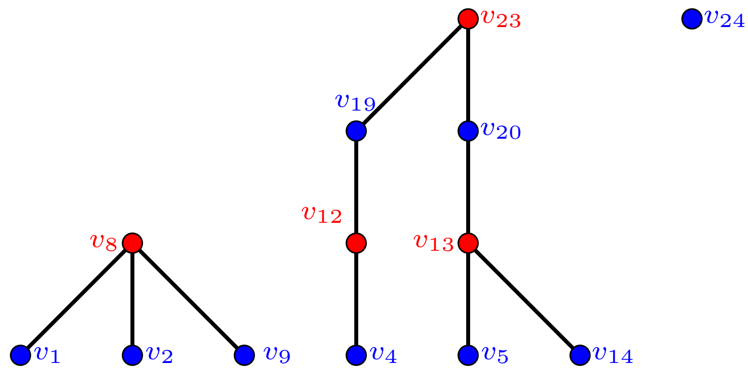


Figure 3.4: T_B from Example 2.8.27

labeling them as T_1, T_2, and T_3 from left to right, we get

$$\frac{R_1}{N_{odd}(T_1)} = \frac{k[v_1, v_2, v_9]}{\langle v_1 v_2 v_9 \rangle}$$

$$\frac{R_2}{N_{odd}(T_2)} = \frac{k[v_4, v_5, v_{14}, v_{19}, v_{20}]}{\langle v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20} \rangle}$$

$$\frac{R_3}{N_{odd}(T_3)} = \frac{k[v_{24}]}{\langle v_{24} \rangle}.$$

Also, we have

$$\frac{R}{N_{odd}(T_B)} = \frac{k[v_1, v_2, v_9, v_4, v_5, v_{14}, v_{19}, v_{20}, v_{24}]}{\langle v_1 v_2 v_9, v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20}, v_{24} \rangle}.$$

Thus we get

$$\frac{R}{N_{odd}(T_B)} = \frac{k[v_1, v_2, v_9, v_4, v_5, v_{14}, v_{19}, v_{20}, v_{24}]}{\langle v_1 v_2 v_9, v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20}, v_{24} \rangle}$$

$$= \frac{k[v_1, v_2, v_9]}{\langle v_1 v_2 v_9 \rangle} \otimes_k \frac{k[v_4, v_5, v_{14}, v_{19}, v_{20}]}{\langle v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20} \rangle} \otimes_k \frac{k[v_{24}]}{\langle v_{24} \rangle}$$

$$= \frac{R_1}{N_{odd}(T_1)} \otimes_k \frac{R_2}{N_{odd}(T_2)} \otimes_k \frac{R_3}{N_{odd}(T_3)}.$$

The only minimal V_3-TD-set for T_1 and T_3 is the empty set since $V_3(T_1) = \emptyset = V_3(T_3)$, and the minimal V_3-TD-sets for T_2 are $\{v_{19}\}$ and $\{v_{20}\}$. Thus by Theorem 3.2.11, we get

$$\text{type}(R_1/N_{odd}(T_1)) = 1 \qquad \text{type}(R_2/N_{odd}(T_2)) = 2 \qquad \text{type}(R_3/N_{odd}(T_3)) = 1.$$

Therefore, the type of $R/N_{odd}(T_B)$ is

$$\text{type}(R/N_{odd}(T_B)) = \text{type}(R_1/N_{odd}(T_1)) \cdot \text{type}(R_2/N_{odd}(T_2)) \cdot \text{type}(R_3/N_{odd}(T_3)) = 2.$$

Here is the main result of this thesis. It shows how to compute the Cohen-Macaulay type of $R/N(T)$ for unmixed trees using only graph-theoretic information about T.

Theorem 3.3.9. *Let $R = k[V]$, $R_B = k[V_{even}(T_B)]$, and $R_R = k[V_{even}(T_R)]$. Let m_B and m_R be the numbers of minimal V_3-TD-sets in T_B and T_R, respectively. The Cohen-Macaulay type of $R/N(T)$ is*

$$\begin{aligned} \mathit{type}(R/N(T)) &= \mathit{type}(R_B/N_{odd}(T_B)) \cdot \mathit{type}(R_R/N_{odd}(T_R)) \\ &= m_B \cdot m_R. \end{aligned}$$

Proof. Let Z_B and Z_R be the maximal regular sequences for $k[V_{even}(T_B)]/N_{odd}(T_B)$ and $k[V_{even}(T_R)]/N_{odd}(T_R)$ computed in Theorem 3.3.7, respectively, and let $Z =$

$Z_B \cup Z_R$. Then we have

$$\frac{R}{N(T)+\langle Z\rangle} = \frac{R}{N_{odd}(T_B)+N_{odd}(T_R)+P_{V_1(T)}+\langle Z\rangle} \qquad \text{(Theorem 3.3.1)}$$

$$= \frac{k[V_{even}(T_B)\cup V_{even}(T_R)\cup V_1(T)]}{N_{odd}(T_B)+N_{odd}(T_R)+P_{V_1(T)}+\langle Z\rangle} \qquad \text{(Lemma 3.3.3)}$$

$$\cong \frac{k[V_{even}(T_B)\cup V_{even}(T_R)]}{N_{odd}(T_B)+N_{odd}(T_R)+\langle Z\rangle}$$

$$= \frac{k[V_{even}(T_B)]}{N_{odd}(T_B)+\langle Z_B\rangle} \otimes_k \frac{k[V_{even}(T_R)]}{N_{odd}(T_R)+\langle Z_R\rangle}.$$

Note that we have

$$\dim\left(\frac{k[V_{even}(T_B)]}{N_{odd}(T_B)+\langle Z_B\rangle}\right) = 0 = \dim\left(\frac{k[V_{even}(T_R)]}{N_{odd}(T_R)+\langle Z_R\rangle}\right)$$

by Theorem 3.3.7 and Fact 2.7.5. Hence we get

$$\dim\left(\frac{R}{N(T)+\langle Z\rangle}\right) = 0$$

by Lemma 2.6.30, so

$$\operatorname{depth}\left(\frac{R}{N(T)+\langle Z\rangle}\right) = 0.$$

Let $\mathcal{D}$ be the set of minimal TD-sets of T, let $\mathcal{D}_B$ and $\mathcal{D}_R$ be the sets of minimal odd TD-sets of T_B and T_R, respectively. Then [7, Lemma 4.2.15] and Corollary 3.1.6 gives us that $D \in \mathcal{D}$ if and only if $D = D_B \cup D_R \cup V_1(T)$ for some $D_B \in \mathcal{D}_B$ and $D_R \in \mathcal{D}_R$; note that the sets D_B, D_R, and $V_1(T)$ are pairwise disjoint by Lemma 3.3.3 since $D_B \subseteq V_{even}(T_B)$ and $D_R \subseteq V_{even}(T_R)$. Fix $D \in \mathcal{D}$, $D_B \in \mathcal{D}_B$, and $D_R \in \mathcal{D}_R$

such that $D = D_B \cup D_R \cup V_1(T)$. Then we have

$$\dim\left(\frac{R}{N(T)}\right) = |V(T)| - |D| \qquad \text{(Facts 2.8.16, 2.6.15)}$$

$$= |V_{even}(T_B) \cup V_{even}(T_R) \cup V_1(T)| - |D| \qquad \text{(Lemma 3.3.3)}$$

$$= |V_{even}(T_B) \cup V_{even}(T_R) \cup V_1(T)| - |D_B \cup D_R \cup V_1(T)|$$

$$= |V_{even}(T_B)| + |V_{even}(T_R)| - |D_B| - |D_R|$$

$$= \dim\left(\frac{R_B}{N_{odd}(T_B)}\right) + \dim\left(\frac{R_R}{N_{odd}(T_R)}\right)$$

$$= \operatorname{depth}\left(\frac{R_B}{N_{odd}(T_B)}\right) + \operatorname{depth}\left(\frac{R_R}{N_{odd}(T_R)}\right) \qquad \text{(Corollary 3.3.6)}$$

$$= |Z_B| + |Z_R|$$

$$= |Z|.$$

Thus Z is a maximal $R/N(T)$-regular sequence by Facts 2.7.5 and 2.9.16. Writing $V_B := V_{even}(T_B)$ and $V_R := V_{even}(T_R)$, we get

$$\operatorname{type}(R/N(T))$$

$$= \operatorname{type}\left(\frac{R}{N(T) + \langle Z\rangle}\right) \qquad \text{(Fact 2.7.7)}$$

$$= \text{type}\left(\frac{k[V_B]}{N_{odd}(T_B)+\langle Z_B\rangle} \otimes_k \frac{k[V_R]}{N_{odd}(T_R)+\langle Z_R\rangle}\right)$$

$$= \dim_k\left(\text{Ext}^0_{k[V_B\cup V_R]}\left(k, \frac{k[V_B]}{N_{odd}(T_B)} \otimes_k \frac{k[V_R]}{N_{odd}(T_R)}\right)\right) \qquad \text{(Example 2.4.3)}$$

$$= \dim_k\left(\text{Ext}^0_{k[V_B]}\left(k, \frac{k[V_B]}{N_{odd}(T_B)+\langle Z_B\rangle}\right) \otimes_k \ \text{Ext}^0_{k[V_R]}\left(k, \frac{k[V_R]}{N_{odd}(T_R)+\langle Z_R\rangle}\right)\right)$$

(Fact 2.4.5)

$$= \dim_k\left(\text{Ext}^0_{k[V_B]}\left(k, \frac{k[V_B]}{N_{odd}(T_B)+\langle Z_B\rangle}\right)\right) \cdot \dim_k\left(\text{Ext}^0_{k[V_R]}\left(k, \frac{k[V_R]}{N_{odd}(T_R)+\langle Z_R\rangle}\right)\right)$$

$$= \text{type}\left(\frac{k[V_B]}{N_{odd}(T_B)+\langle Z_B\rangle}\right) \cdot \text{type}\left(\frac{k[V_R]}{N_{odd}(T_R)+\langle Z_R\rangle}\right)$$

$$= \text{type}(R/N_{odd}(T_B)) \cdot \text{type}(R/N_{odd}(T_R)) \qquad \text{(Fact 2.7.7)}$$

$$= m_B \cdot m_R \qquad \text{(Theorem 3.3.7)}$$

as desired. □

We end this chapter with some examples applying Theorem 3.3.9.

Example 3.3.10. Let T be the tree from Example 2.8.26. Consider its blue and red interior graphs T_B and T_R in Figure 3.5. The minimal V_3-TD-sets of T_B are $\{u_3, u_5\}$ and $\{u_4\}$. The minimal V_3-TD-sets of T_R are $\{u_1\}$ and $\{u_2\}$. Therefore, we have $\text{type}(R/N(T)) = 2 \cdot 2 = 4$.

Example 3.3.11. Let T be the tree in Example 2.8.27 and let $R = k[V(T)]$. We

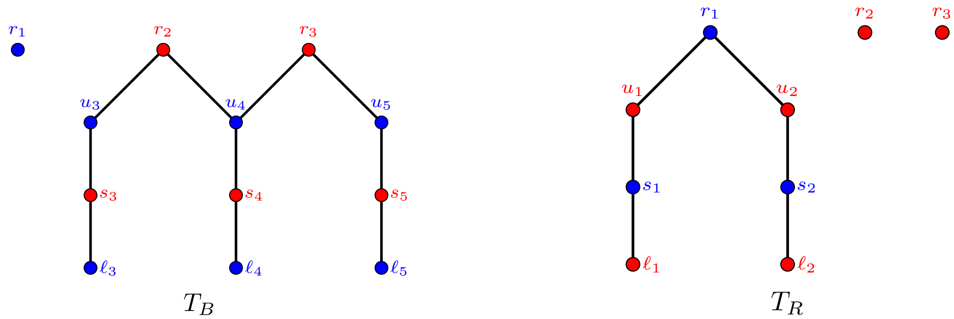


Figure 3.5: T_B and T_R from Example 2.8.26

have

$$\text{type}(k[V_{even}(T_B)]/N_{odd}(T_B)) = 2$$

from Example 3.3.8. Since T_B and T_R are isomorphic graphs, we get

$$k[V_{even}(T_B)]/N_{odd}(T_B) \cong k[V_{even}(T_R)]/N_{odd}(T_R)$$

which gives

$$\text{type}(k[V_{even}(T_R)]/N_{odd}(T_R)) \cong \text{type}(k[V_{even}(T_B)]/N_{odd}(T_B)) = 2.$$

Thus we get

$$\begin{aligned}\text{type}(R/N(T)) &= \text{type}(k[V_{even}(T_B)/N_{odd}(T_B)]) \cdot \text{type}(k[V_{even}(T_R)]/N_{odd}(T_R)) \\ &= 2 \cdot 2 = 4.\end{aligned}$$

Chapter 4

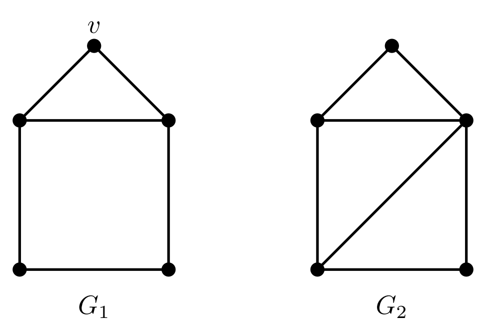
Future Work

4.1 Other Families of Graphs

We first define two different types of graphs.

Definition 4.1.1. Let $G = (V, E)$ be a finite simple graph. Then G is *bipartite* if G has a 2-coloring. An *induced cycle* of G is an induced subgraph of G that forms a cycle graph. And G is *chordal* if every induced cycle of G is a 3-cycle.

Example 4.1.2. Every tree is bipartite and chordal (vacuously). Consider the following graphs.



Then G_1 is not chordal since the induced cycle formed by all the vertices but v is a 4-cycle. On the other hand, G_2 is chordal.

The notion of "edge ideal" (instead of taking the open neighborhood of a vertex as the monomial generator, take edges) was first introduced by Villarreal in [16]. He investigated trees whose corresponding edge ideals are Cohen-Macaulay. Later, people extended his work by looking at different family of graphs such as bipartite graphs and chordal graphs [4, 9]. Similar to the edge ideal case, we can ask a similar question:

Question 4.1.3. What are the characterizations of unmixed bipartite graphs and unmixed chordal graphs?

Once we have an answer to Question 4.1.3, we can ask the Cohen-Macaulayness of such families of graphs as well.

In general, answering Question 4.1.3 is not obvious. Hence answering Question 4.1.4 is not trivial as well.

Question 4.1.4. Are the families of graphs from Question 4.1.3 Cohen-Macaulay?

4.2 Sequential Cohen-Macaulayness

An alternative question we can ask is to characterize a family of trees, chordal graphs, or bipartite graphs whose open neighborhood ideals are "sequentially Cohen-Macaulay." This is due to the following theorem.

Theorem 4.2.1 ([5, Lemma 3.6])**.** *Let I be a squarefree monomial ideal in $R = k[x_1, ..., x_n]$ where k is a field. Then R/I is Cohen-Macaulay if and only if R/I is sequentially Cohen-Macaulay and I is unmixed.*

Since we are interested in the open neighborhood ideals that are Cohen-Macaulay, we could look at the ones that are sequentially Cohen-Macaulay. The following is the definition of sequentially Cohen-Macaulay modules.

Definition 4.2.2. Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k. Let M be a graded R-module. We say M is *sequentially Cohen-Macaulay* (over R) if there exists a finite filtration of graded R-submodules

$$0 = M_1 \subset M_2 \subset \cdots \subset M_r = M$$

such that for all $i \in [r]$, M_i/M_{i-1} is Cohen-Macaulay, and

$$\dim_k(M_2/M_1) < \dim_k(M_3/M_2) < \cdots < \dim_k(M_r/M_{r-1}) .$$

A graded ideal $I \leq R$ is *sequentially Cohen-Macaulay* if the quotient R/I is sequentially Cohen-Macaulay as an R-module.

We state some results about sequentially Cohen-Macaulay edge ideals.

Theorem 4.2.3 ([15, Theorem 2.13])**.** *Let G be a chordal graph and let $I(G)$ be its edge ideal. Then $I(G)$ is sequentially Cohen-Macaulay, hence $I(G)$ is sequentially Cohen-Macaulay if G is a forest.*

Theorem 4.2.4 ([15, Theorem 3.10])**.** *Let G be a bipartite graph, $I(G)$ its edge ideal, and Δ_G the "independence complex" of G. Then $I(G)$ is sequentially Cohen-Macaulay if and only if Δ_G is shellable.*

This leads to the following question.

Question 4.2.5. (When) are the open neighborhood ideals of trees, chordal graphs, and bipartite graphs sequentially Cohen-Macaulay?

4.3 Minimal Free Resolutions

Consider the following theorem.

Theorem 4.3.1 (Hilbert's Syzygy Theorem)**.** *Let $R = k[X_1, \ldots, X_d]$ where k is a field.*

(1) If $I = \langle f_1, \ldots, f_{\beta_1} \rangle \leq R$, then there exists an exact sequence

$$0 \longrightarrow R^{\beta_d} \xrightarrow{\partial_d} \cdots \xrightarrow{\partial_{\beta_3}} R^{\beta_2} \xrightarrow{\partial_2} R^{\beta_1} \xrightarrow{\partial_1} R \xrightarrow{\nu} R/I \longrightarrow 0$$

where $\partial_1 = (f_1 \ \cdots \ f_{\beta_1})$ and $\beta_i \in \mathbb{N}$ for all $i \in [d]$. The sequence above is called an (augmented) free resolution *of R/I over R.*

(2) If f_i is homogeneous for all $i \in [d]$ in (1), then this resolution can be built minimally and the β_j's are independent of the choice of minimal free resolution. The integer β_j is the jth Betti number *of R/I over R.*

From Theorem 4.3.1, it turns out that the dth (highest degree) Betti number is the Cohen-Macaulay type of R/I over R . Hence computing the minimal free resolution of open neighborhood ideals of unmixed trees can generalize Theorem 3.3.9. There is a general formula for finding the Betti numbers of R/I when I is a squarefree monomial ideal known as Hochster's formula. But we would like to find a graph theoretical way to compute the Betti numbers just like Theorem 3.3.9.

Question 4.3.2. How to compute the minimal free resolutions (hence the Betti numbers) of open neighborhood ideals of unmixed trees directly from the trees.

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